

# ON SOME TWO-DIMENSIONAL INTEGRAL EQUATIONS WITH HOMOGENEOUS KERNELS

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**Abstract**

**Full Text**

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**MATHEMATICS**

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## **ON SOME TWO-DIMENSIONAL INTEGRAL EQUATIONS WITH HOMOGENEOUS KERNELS**

*(Presented by Academician A. N. Tikhonov on 30 V 1969)*

We shall use complex notation for points of the plane  $E_2$  and consider the equation

$$f(z) = \iint_{E_2} \theta(z, \zeta) f(\zeta) dS_\zeta + g(z), \quad |z| \leq R. \quad (1)$$

The kernel  $\theta(z, \zeta)$  satisfies the homogeneity condition

$$\theta(tz, t\zeta) = |t|^{-2} \theta(z, \zeta) \quad (2)$$

for any complex  $t$ , and the summability condition

$$Q_\beta = \iint_{E_2} |\theta(1, \zeta)| |\zeta|^{-\beta} dS_\zeta < +\infty. \quad (3)$$

Taking  $t$  real and  $t > 0$ , from (2) we obtain the usual homogeneity condition of order  $-2$ , and putting  $t = e^{i\alpha}$ ,  $0 \leq \alpha < 2\pi$ , the condition of invariance with respect to rotations <sup>(1,2)</sup>. In view of (2),  $\theta(z, \zeta)$  is defined on the whole plane.

The number  $\beta$  in (3) determines the weight classes of solutions and free terms <sup>(1,2)</sup>. If  $\Phi(z) \in M, C, L^p$ , then by  $M_\beta, C_\beta, L_\beta^p$  are denoted the classes of functions

$$\varphi(z) = |z|^{-\beta} \Phi(z), \quad \|\varphi\|_\beta = \|\Phi\|.$$

Under condition (3), the integral operator in (1) is bounded in the spaces  $M_\beta, C_\beta$ , and  $L_{\beta-2/p}^p$  <sup>(2)</sup>. With respect to the last of these spaces it should be noted that the other summability condition required for it reduces to (3) by the substitution  $\nu = 1/\xi$ , using the homogeneity (2). Examples of functions satisfying conditions (2), (3) are the kernel  $1/\zeta(\zeta - z)$ , which plays a fundamental role in the singular

case of the theory of generalized analytic functions, and  $|z|^{-\alpha}|\zeta - z|^{\alpha-2}$ ,  $0 < \alpha < 2$ , which has an analogous significance for second-order differential equations with singular coefficients <sup>(3)</sup>.

In order not to burden the exposition, we pass to unweighted classes of functions and restrict ourselves to the space  $M$ . Then instead of (1) we obtain

$$F(z) = \iint_{|\zeta| \leq R} T(z, \zeta) F(\zeta) dS_\zeta + G(z), \quad |z| \leq R, \quad (4)$$

where  $T(z, \zeta) = |z/\zeta|^\beta \theta(z, \zeta)$ , and condition (3) reduces simply to the condition of summability of the kernel  $T(1, \zeta)$  over the whole plane.

After the substitution  $\zeta = \sigma z$ , (4) takes the form

$$F(z) = \iint_{|\sigma||z| \leq R} T(1, \sigma) F(\sigma z) dS_\sigma + G(z). \quad (5)$$

The method of solution consists in passing to polar coordinates

$$z = r e^{i\varphi}, \quad \zeta = \rho e^{i\theta}, \quad \sigma = \tau e^{i\alpha},$$

we construct the solution  $F(z) \equiv F(r, \varphi)$  from its Fourier coefficients in  $\varphi$

$$F_k(r) = \frac{1}{2\pi} \int_0^{2\pi} F(r, \varphi) e^{-ik\varphi} d\varphi.$$

Multiplying (5) by  $\frac{1}{2\pi} e^{-ik\varphi}$ , integrating and using (2), we obtain

$$F_k(r) = \int_0^R T_k(r, \rho) F_k(\rho) d\rho + G_k(r), \quad k = 0, \pm 1, \pm 2, \dots, \quad (6)$$

where

$$T_k(r, \rho) = \frac{1}{r} \Omega_k \left( \frac{\rho}{r} \right), \quad (7)$$

$$\Omega_k(\tau) = \tau \int_0^{2\pi} T(1, \tau e^{i\alpha}) e^{ik\alpha} d\alpha = -i\tau \int_{|t|=1} T(1, \tau t) t^{k-1} dt.$$

The two-dimensional equation (4) with a kernel homogeneous of order  $-2$  has been reduced to the diagonal system (6) of one-dimensional integral equations with kernels homogeneous of degree  $-1$ . Equations of precisely this type were

studied in (1). The summability conditions for the kernels  $T_k(r, \rho)$  will be satisfied by virtue of (3):

$$Q_k = \int_0^\infty |T_k(1, u)| du = \int_0^\infty u du \cdot \left| \int_0^{2\pi} T(1, ue^{i\alpha}) e^{ik\alpha} d\alpha \right| \leq Q(\beta). \quad (8)$$

Since, by virtue of (3), the function  $uT(1, ue^{i\alpha})$  is summable in  $\alpha$  for almost all  $u$ , by the Riemann-Lebesgue theorem (6)

$$\lim_{|k| \rightarrow \infty} \int_0^{2\pi} uT(1, ue^{i\alpha}) e^{ik\alpha} d\alpha = 0.$$

Applying Lebesgue's theorem on passage to the limit under the integral sign, we obtain  $\lim_{|k| \rightarrow \infty} Q_k = 0$ .

The determining functions for equations (6) are the Fourier transforms of the kernels

$$\mathcal{H}_k(x) = \int_0^\infty u^{ix} T_k(1, u) du = \iint_{E'_2} T(1, \sigma) |\sigma|^{ix} e^{ik\alpha} dS_\sigma. \quad (9)$$

From (8), (9) it is clear that

$$\max_{-\infty < x < \infty} |\mathcal{H}_k(x)| \leq Q_k.$$

Taking into account that the numbers  $Q_k$  give upper estimates for the norms of the operators from (6), we conclude: there exists a number  $N > 0$  such that, for  $|k| \leq N$ , equations (6) are uniquely solvable for arbitrary free terms. The choice of  $N$  may be made from the condition  $Q_k < 1$ ,  $|k| > N$ , or

$$\max_{-\infty < x < \infty} |\mathcal{H}_k(x)| < 1, \quad |k| > N. \quad (10)$$

Denote

$$\mathcal{G}_k(x) = 1 - \mathcal{H}_k(x), \quad \varkappa_k = -\text{Ind } \mathcal{G}_k(x) = -\frac{1}{2\pi} \{\arg \mathcal{G}_k(x)\}_{-\infty}^{\infty}, \quad (11)$$

$$\mathcal{G}(x) = \prod_{|k| \leq N} \mathcal{G}_k(x), \quad \varkappa_+ = \sum_{\varkappa_k > 0} \varkappa_k, \quad \varkappa_- = -\sum_{\varkappa_k < 0} \varkappa_k.$$

According to (1), the normality conditions for equations (6) will be  $\mathcal{G}_k(x) \neq 0$ ,  $-\infty < x < \infty$ ,  $\varkappa_k$  are their indices, while (10) shows that for

$|k| > N$  the conditions will be satisfied and  $\chi_k = 0$ . Therefore one may say that the two-dimensional equation (4), and together with it also (1), is equivalent to the finite diagonal system (6). Solutions of the homogeneous equation (4) are obtained directly from solutions of the homogeneous equations (6) by the formula  $F(z) \equiv F(r, \varphi) = F_k(r)e^{ik\varphi}$ . As for the nonhomogeneous equation (4), here, after finding solutions of the system (6), we must also show that they indeed form the Fourier coefficients of a function from  $M$ . We shall give an indirect proof of this fact.

Let  $M_N$  be the subclass of those functions from  $M$  for which  $F_k(r) \equiv 0$  when  $|k| > N$ , and let  $M^N$  be the subclass of functions for which, conversely,  $F_k(r) \equiv 0$  when  $|k| \leq N$ . It is clear that  $M_N$  and  $M^N$  are closed in  $M$  and form Banach spaces whose direct sum is  $M$ .

Equation (4) splits into two:

$$F_N = TF_N + G_N, \quad F_N, G_N \in M_N; \quad (12_1)$$

$$F^N = TF^N + G^N, \quad F^N, G^N \in M^N. \quad (12_2)$$

We shall prove that (12<sub>2</sub>) is uniquely invertible on  $M^N$ . It is not hard to see that the kernel from (4),  $T(1, \sigma) = T(1, \tau e^{i\alpha})$ , can be approximated in  $L$  by functions of the form

$$K_N(\tau, \alpha) = - \sum_{|k| \leq N} w_k(\tau) e^{ik\alpha}.$$

Since integrals with such kernels are annihilating operators on  $M^N$ , there  $T = T - K_N$ , and hence, for sufficiently large  $N$ ,  $\|T\|_{M^N} < 1$ .

As for (12<sub>1</sub>), it splits into the finite part of system (6), where  $|k| \leq N$ .

**Theorem 1.** *Let, in equation (1), the kernel satisfy the homogeneity conditions (2), the summability condition (3), and the normality condition  $\mathcal{G}(x) \neq 0$ ,  $-\infty < x < \infty$ , where  $\mathcal{G}(x)$  (as well as the numbers  $\chi_+, \chi_-$ ) is given by formulas (11), and let the functions  $f(z), g(z) \in M_\beta, C_\beta$ , or  $L_{\beta-2/p}^p$ ,  $p \geq 1$ .*

*Then the homogeneous equation (1) has  $\chi_+$  linearly independent solutions, and for solvability of the nonhomogeneous equation it is necessary and sufficient that the  $\chi_-$  solvability conditions*

$$\iint_{|z| \leq R} g(z)\omega(z) dS_z = 0, \quad (13)$$

*hold, where  $\omega(z)$  are solutions of the transposed homogeneous equation from  $M_{2-\beta}, C_{2-\beta}^0$ , or  $L_{2-\beta-2/p}^p$ ,  $p \geq 1$ , respectively.*

All the assertions except the last have been proved by us. If  $\psi_k(r)$  are solutions of the homogeneous equation transposed to (6), then the solutions of the homogeneous equation transposed to (1) have the form

$$\omega(z) = \frac{\psi_k(r)}{r} e^{ik\varphi}.$$

If  $\psi_k(r) \in M_{1-\beta}$ , then  $\omega(z) \in M_{2-\beta}$ .

Writing out (13), we have

$$\int_0^R \psi_k(r) dr \int_0^{2\pi} e^{ik\varphi} G(r, \varphi) d\varphi = 0.$$

Substituting here  $G = G_N + G^N$ , we see that for  $G^N$  the condition is automatically satisfied. As for

$$G_N = \sum_{|k| \leq N} G_k(r) e^{ik\varphi},$$

it gives

$$\int_0^R G_k(r) \psi_k(r) dr = 0.$$

As was shown in (1), these conditions are necessary and sufficient for solvability of (1). The theorem is proved.

**Remark.** Let  $\Pi$  denote the strip  $0 \leq v < 2\pi$ ,  $-\infty < u < \infty$ , and let  $\Pi_+$  be the half-strip  $u \geq 0$ ; suppose the kernel  $K(w)$  is given in the whole plane of  $E_2$ , is  $\pi$ -periodic,  $K(w + 2n\pi i) = K(w)$ , and summable in the strip

$$\iint_{\Pi} |K(u + iv)| du dv < +\infty.$$

The equation considered is

$$f(z) = \iint_{\Pi_+} K(\zeta - z) f(\zeta) dS_{\zeta} + g(z), \quad z \in \Pi_+,$$

where  $f(z), g(z) \in M(\Pi_+)^*$ . Then for equation (13) a theorem analogous to Theorem 1 is valid.

The idea of the proof is that, putting  $z = e^{-s}$ ,  $\zeta = e^{-\sigma}$  in (1), we transform the disk into a half-plane, and from a homogeneous kernel obtain a difference kernel.

The result obtained differs from the known results [5] for multidimensional equations with difference kernels in that here there is an index  $\varkappa = \varkappa_+ - \varkappa_-$ .

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\* One may also consider other classes of type  $C$ ,  $L^p$ , and their subspaces, obtained from the classes for the corresponding one-dimensional equations (4).

*Note: Figure translations are in progress. See original paper for figures.*

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