CONSTRUCTION OF SEPARATRIX SURFACES AND REGIONS OF STABILITY IN THE PHASE SPACES OF NONLINEAR SYSTEMS

MATHEMATICS

1970

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Abstract

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UDC 517.93

MATHEMATICS

Academician B. N. PETROV, R. A. NELEPIN, B. M. SHAMRIKOV

CONSTRUCTION OF SEPARATRIX SURFACES AND REGIONS OF STABILITY IN THE PHASE SPACES OF NONLINEAR SYSTEMS

1. Consider the system

$$\dot{\eta} = A\eta + bf(\sigma), \qquad \sigma = c'\eta, \tag{1}$$

where η is a column of n variables η_k; $A = (a_{ki})$ is an $n \times n$ matrix; b, c are columns of n constants b_k, c_k (the prime denotes transposition); $f(\sigma)$ is a scalar function; $\dot{} = d/dt$; t is the independent variable (time).

By means of the transformation

$$\eta = Kx \tag{2}$$

we pass from system (1) to the system

$$\dot{x} = \Lambda x + ef(\sigma), \qquad \sigma = \gamma' x. \tag{3}$$

Here x is a column of n variables x_i; $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$; $e = (1, \dots, 1)$ is an n-dimensional column; γ' is a row of n quantities γ_i;

$$\gamma_i = -[D'(\lambda_i)]^{-1} \sum_{k=1}^{n} c_k N_k(\lambda_i); \qquad N_i(\lambda_k) = \sum_{\alpha=1}^{n} b_\alpha D_{\alpha i}(\lambda_k), \quad i = 1, \dots, n; \tag{4}$$

λ_i are the roots of the equation $D(\lambda) \equiv \det[a_{ki} - \delta_{ki}\lambda] = 0$; $D'(\lambda) = dD(\lambda)/d\lambda$; $D_{\alpha i}(\lambda_k)$ is the algebraic cofactor of the element of row α, column i, of the determinant $D(\lambda_k)$. The matrix K is nonsingular under the conditions indicated in (^{1}).

Let a_{ki}, b_k be prescribed numbers, and c_k parameters. In the n-dimensional space C_n of parameters $c_1, \dots, c_n$, define the revealing sections $G_2^{(s,r)}$—planes of dimension 2—by the equations

$$\sum_{k=1}^{n} c_k N_k(\lambda_i) = \delta_{is} A_s + \delta_{ir} A_r, \qquad i = 1, \dots, n, \tag{5}$$

where δ_{ij} is the Kronecker symbol, A_s, A_r are real (if λ_s, λ_r are real) or complex conjugate (if λ_s, λ_r are complex conjugate), and otherwise arbitrary constants. Under the conditions of the section (5) we have $\gamma_i = 0$ $(i \neq s, r)$, and equations (3) for x_s, x_r, σ form an independent subsystem T, from which we find $\sigma(t)$; after this, equations (3) for x_i $(i \neq s, r)$ are integrated as linear inhomogeneous equations.

2. Consider the space X of the variables x_k; since a pair of equations with complex $x_i, x_{i+1} = \bar{x}_i \pm j\bar{x}_{i+1}$ corresponds to a pair of equations with real $\bar{x}_i, \bar{x}_{i+1}$, by the space X we shall mean the Euclidean space of the variables $x_1, \dots, \bar{x}_i, \bar{x}_{i+1}, \dots, x_n$.

Under the conditions of the section $G_2^{(s,r)}$, the phase portrait on the plane with Cartesian coordinates x_s, x_r of the subsystem T may be regarded as the projection of the phase portrait in the n-dimensional space X of the complete system (3) onto the plane x_s, x_r of this space. To a limit cycle Γ in the plane x_s, x_r there corresponds a separatrix surface S_Γ of dimension $n-1$ in the space X, representing a cylindrical surface parallel to the axes x_i $(i \neq s, r)$ and forming, in its intersection with the plane x_s, x_r, a closed-

curve Γ. Analogously, the separatrix surfaces $S_c, S_{\text{o.p}}$, generated by separatrices and rest segments of the plane x_s, x_r, are defined.

Let there be an equation of a limit cycle in the plane x_s, x_r

$$S(x_s, x_r) = 0. \tag{6}$$

Then the equation of the separatrix surface S_Γ in the space of the variables $\eta_1, \dots, \eta_n$ will be

$$S\left[(\det K)^{-1} \sum_{i=1}^{n} K_{is}\eta_i;\ (\det K)^{-1} \sum_{i=1}^{n} K_{ir}\eta_i\right] = 0, \tag{7}$$

where K_{ij} is the algebraic cofactor of the element in row i, column j of the determinant of the matrix K.

3. Let an unstable limit cycle Γ bound the region of asymptotic stability of the equilibrium state of the subsystem T in the plane x_s, x_r, and let $\operatorname{Re}\lambda_i < 0$ $(i \neq s, r)$; then surface (7) in the section $G_2^{(s,r)}$ serves as the

exact boundary of the domain of attraction of the equilibrium state in the n-dimensional phase space of system (1).

Construct the Lyapunov function

$$V(x_1, \dots, x_n) = \sum_{\substack{i=1 \\ i \neq s,r}}^{n} h_i x_i^2 + h_r \xi_r^2 + h_s \xi_s^2, \tag{8}$$

where $h_r \xi_r^2 + h_s \xi_s^2 = \text{const}$ is the equation of an ellipse approximating the cycle Γ and determined by the method of harmonic linearization; ξ_s, ξ_r are linear functions of x_s, x_r, determined by reducing the equation of the ellipse to its principal axes; h_i are positive constants.

Using Sylvester' s inequalities, we determine the values of the parameters c_k in $G_2^{(s,r)}$ for which $\dot{V} < 0$ in a neighborhood of the origin.

Consider the surfaces

$$F(x_1, \dots, x_n) \equiv \dot{V}(x_1, \dots, x_n) = 0; \tag{9}$$

$$\Phi(x_1, \dots, x_n) \equiv \sum_{\substack{i=1 \\ i \neq s,r}}^{n} h_i x_i^2 + h_r \xi_r^2 + h_s \xi_s^2 - V_i = 0, \tag{10}$$

where V_i is some number. Define the domain of attraction of the equilibrium state Q_i as the largest domain bounded by surface (10) and wholly inscribed in the domain bounded by surface (9). In particular, for a smooth function $f(\sigma)$, the boundary of the domain Q_i is described by equation (10) for $V_i = V_i^*$, where V_i^* is the smallest of the numbers V_i determined by solving the system

$$\left(\frac{\partial \Phi}{\partial x_i}\right)_* \left(\frac{\partial F}{\partial x_i}\right)_*^{-1} = \text{const}, \qquad i = 1, \dots, n; \tag{11}$$

$$F(x_1^*, \dots, x_n^*) = \Phi(x_1^*, \dots, x_n^*) = 0.$$

Under the conditions of the section $G_2^{(s,r)}$, system (11) has a solution as $h_i \to 0$ $(i \neq s, r)$ and determines the domain of attraction of the equilibrium state as the interior of an n-dimensional cylinder.

4. Consider a neighborhood of the section $G_2^{(s,r)}$, where $\gamma_i = \varepsilon_i \leq \varepsilon$ $(i \neq s, r)$, $\varepsilon > 0$ is a small number. By an orthogonal transformation of variables

$$x = Lz \tag{12}$$

we reduce system (3) to the form

$$\dot{z} = Dz + gf(\sigma), \qquad \sigma = \beta' z. \tag{13}$$

Here $L = (l_{ij})$, $D = (d_{ij})$ are $n \times n$ matrices; g is an n-dimensional column; $\beta' = (0, \dots, \beta_s, \dots, \beta_r, \dots, 0)$ is an n-dimensional row; $\beta_s \beta_r^{-1} = \gamma_s \gamma_r^{-1}$. Elements the matrices L' satisfy the relations

$$\sum_{j=1}^{n} l_{jk} \gamma_j = 0, \qquad k = 1, \dots, s-1,\ s+1, \dots,\ r-1,\ r+1, \dots,\ n;$$

$$\sum_{j=1}^{n} l_{ij} l_{jk} = \delta_{jk}, \qquad j, k = 1, \dots, n, \tag{14}$$

and are determined up to $\frac{1}{2}(n^2 - 3n + 2)$ arbitrary constants. These constants can be chosen so that the numbers $|d_{ij}|$ $(i \neq j)$ are sufficiently small, and from the conditions $\varepsilon_i \to 0$ $(i \neq s, r)$ it follows that $d_{ij} \to 0$ $(i \neq j)$. By the method of harmonic linearization we determine a cycle Γ in the space Z of the variables $z_1, \dots, z_n$ and consider the ellipse generated by it in the Cartesian coordinates z_s, z_r. After this we determine the function V by formula (8), in which we replace x_i by z_i, and repeat the exposition of item 3. The solution of system (11) will now determine the surface bounding a certain elongated body. The choice of h_i $(i \neq s, r)$ makes it possible to construct a family of regions Q_i; the desired region will be their union. Transformations (12) and (2) make it possible to recalculate this region into the space of the variables $\eta_1, \dots, \eta_n$ of the original system (1).

The method considered for constructing regions of stability finds application in the design of control systems for moving objects (^{2}).

Received18 XII 1969

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