ON BICOMPACTA WITH NONCOINCIDING DIMENSIONS

1970

SovietRxiv

Abstract

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UDC 513.831

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(Presented by Academician P. S. Aleksandrov, 18 XI 1969)

This note adjoins our joint note with I. K. Lifanov ([1]) (see the beginning of ([1])).

1. We generalize the construction described in Sections 3 and 4 of note ([1]).

(a) Suppose we are given: a bicompactum X, a pair of its closed subsets F_1 and $F_2 \supset F_1$, bicompacta Φ_1 and Φ_2, a continuous mapping f of the bicompactum Φ_1 onto the bicompactum F_1, and a continuous mapping g of the bicompactum Φ_0 onto the bicompactum Φ_1.

Consider the product $X \times \Phi_1$. Denote by p the natural projection of $X \times \Phi_1$ onto X. The bicompactum Φ_1 may be identified with the graph $\Psi \subset X \times \Phi_1$ of the mapping $f : \Phi_1 \to F_1 \subset X$. Denote by q the mapping of the product $X \times \Phi_0$ onto the product $X \times \Phi_1$ which assigns to the point (x, φ_0), $x \in X$, $\varphi_0 \in \Phi_2$, the point $(x, g(\varphi_0))$. Consider the bicompactum $q^{-1}p^{-1}F_2 = F_2 \times \Phi_0 \subset X \times \Phi_0$ and its decomposition v. The elements of v are: 1) the individual points of the set $F_2 \times \Phi_0 \setminus q^{-1}\Psi$, and 2) the sets $q^{-1}(x, \varphi_1)$ for $(x, \varphi_1) \in \Psi \equiv \Phi_1$. We denote the decomposition space v by $E = E(X, F_2, F_1, \Phi_1, \Phi_0, f, g)$. Denote by v the natural mapping of the bicompactum $F_2 \times \Phi_0$ onto E. Denote by μ the mapping of the bicompactum E into $X \times \Phi_1$ satisfying the condition $\mu \cdot v = q$. Obviously, on the set $\mu^{-1}\Psi$ the mapping μ is a homeomorphism, and therefore the set $\mu^{-1}\Psi$ may be identified with the bicompactum $\Phi_1 \equiv \Psi$. The mapping $\mathfrak{P} = p \cdot \mu$ will be called the projection of the bicompactum E onto the bicompactum X.

(b) Let a bicompactum X and a system of pairs (F_2^θ, F_1^θ) of its closed subsets F_1^θ and $F_2^\theta \supset F_1^\theta$, $\theta \in \Theta$, be given. Let, in addition, a bicompactum χ be given, and in it a disjoint system of open sets C_θ, $\theta \in \Theta$, each of which decomposes into the disjoint sum of open-closed bicompacta Φ_0^α, $\alpha \in \mathfrak{A}_\theta$. Suppose that for each α a mapping g_α of the bicompactum Φ_0^α onto some bicompactum Φ_1^α, and a mapping f_α of the bicompactum Φ_1^α onto the bicompactum F_1^θ, $\alpha \in \mathfrak{A}_\theta$, $\theta \in \Theta$, are defined. Take the decomposition v of the bicompactum $Y = X \times \chi \setminus \bigcup_\theta (X \setminus F_2^\theta \times C_\theta)$ into the individual points of the set $X \times (\chi \setminus \bigcup_\theta C_\theta)$ and the inverse images of the points of the bicompactum $E_\alpha = E(X, F_2^\theta, F_1^\theta, \Phi_1^\alpha, \Phi_0^\alpha, f_\alpha, g_\alpha)$ under the mapping

$v_\alpha : F_2^\theta \times \Phi_0^\alpha \to E_\alpha$ (see (a)) for all $\alpha \in \mathfrak{A}_\theta$ and $\theta \in \Theta$. We denote the decomposition space v by

$$E = E(X, \chi, \{F_2^\theta, F_1^\theta, \Phi_1^\alpha, \Phi_0^\alpha, f_\alpha, g_\alpha;\ \alpha \in \mathfrak{A}_\theta\},\ \theta \in \Theta). \qquad (*)$$

Denote by v the natural mapping of the bicompactum Y onto the bicompactum E. Denote by p the natural projection of $X \times \chi$ onto X. The mapping $\mathfrak{P} : E \to X$, satisfying the relation $\mathfrak{P} \cdot v = p$, will be called the projection of E onto X. If the bicompacta $v(F_2^\theta \times \Phi_0^\alpha)$ are identified with the bicompacta E_α, then on E_α the projection v coincides with the projection $v_\alpha : E_\alpha \to X$.

Remark 1. a) If the bicompactum Y is the image under a mapping f of the bicompactum X, then below a point $y \in Y$ will often be denoted by one (or several) points of its inverse image $f^{-1}y$. b) Let the bicompactum $Y = \{y\}$ be the image under a finite-to-one mapping f of the bicompactum X, which, in turn, is the product $X_1 \times X_2$ of ordered bicompacta $X_1 = \{x_1\}$ and $X_2 = \{x_2\}$. Then by $+{}^1/_2(y_0, Y, j)$, respectively $-{}^1/_2(y_0, j) = -{}^1/_2(y_0, Y, j)$, $j = 1, 2$, we denote the set

$$\{y :\ y \in f\{(x_1, x_2);\ x_j \geq \max x_j^0, \text{ respectively } \leq \min x_j^0, \text{ for } (x_1^0, x_2^0) \in f^{-1}y_0\}\}.$$

2. Bicompacta T_1 and S_1. Consider ordered bicompacta

$L_k = \{l_k\}$ with such points $a_k \in L_k$ that the first axiom of countability holds at a_k, $k = 0, 1$. Consider the products $L_k \times C_k$, where $C_k = \{c_k\}$ denotes the Cantor perfect set, $k = 0, 1$. By ω_k we denote the decomposition of $L_k \times C_k$ whose elements are pairs of endpoints of intervals adjacent to the Cantor set $a_k \times C_k$, and the individual points not belonging to these pairs. The decomposition space ω_k will be denoted by Γ'_k, and the natural mapping of $L_k \times C_k$ onto Γ'_k by ω_k (as also the corresponding decomposition). By M'_k we denote the set of those points of the bicompactum Γ'_k whose inverse image under the mapping ω_k consists of two points. The set $Q_k^1 = \omega_k(a_k \times C_k)$ is a segment, on which the countable set M'_k is everywhere dense. There exists a homeomorphism h of the segment Q_0^1 onto the segment Q_1^1, for which $hM'_0 = Q_1^1 \setminus M'_1$. The decomposition space of the discrete sum of the bicompacta Γ'_0 and Γ'_1 into the individual points of the set $(\Gamma'_0 \cup \Gamma'_1) \setminus (Q_0^1 \cup Q_1^1)$ and the pairs of points $(x, h(x))$, $x \in Q_0^1$, will be denoted by $\Gamma = \Gamma(L_0, L_1, a_0, a_1)$. The natural mapping of $\Gamma'_0 \cup \Gamma'_1$ onto Γ will be denoted by ω. Let $Q_\Gamma^1 = \omega(Q_0^1 \cup Q_1^1)$, $\Gamma_k = \omega\Gamma'_k$, $M_k = \omega M'_k$, $k = 0, 1$, $N = Q_\Gamma^1 \setminus (M_0 \cup M_1)$.

Mark the following pairs (x, F) of points $x \in \Gamma$ and closed subsets $F \supset x$ in Γ: a) $x \in N$, $F = \Gamma$; b) $x \in M_0 \cup M_1$, $F = +^{1/2}(x, \Gamma_0, 2) \cup +^{1/2}(x, \Gamma_1, 2)$; $x \in M_0 \cup M_1$, $F = -^{1/2}(x, \Gamma_0, 2) \cup -^{1/2}(x, \Gamma_1, 2)$; c) $x = (a_k, c_k) \in M_{\bar{k}}*$, the set F satisfies the following conditions: F is the closure of an open subset of Γ, $F \cap \Gamma_{\bar{k}} = x$, and for some points $l'_k < a_k$ and $l''_k > a_k$ the set $\{(l_k, c_k) : l'_k \leq l_k \leq l''_k\}$ is contained in the interior of F_k, $k = 0, 1$. The set of marked pairs will be denoted by B'.

Consider the bicompacta χ_1 and χ_2 from (1) and establish a one-to-one correspondence: 1) between the set B' and the set Θ of item 5 of remark (1); 2) between the set B' and the set Θ' of item 51 of remark (1). Fix some mapping g of the Cantor perfect set C onto the segment Q^1. Put the bicompactum $T_1 = T_1(L_0, L_1, a_0, a_1)$, respectively $S_1 = S_1(L_0, L_1, a_0, a_1)$, equal to the bicompactum E from formula $(*)$, where $X = \Gamma$; $\chi = \chi_1$, respectively $\chi = \chi_2$; $F_2^\theta = F^\theta$, $F_1^\theta = x_\theta$, where (x_θ, F^θ) is the element of the set B' corresponding to the element θ of the set Θ, respectively Θ'; $\Phi_1^\alpha = Q^1$; $\Phi_0^\alpha = C_\alpha$; $g_\alpha = g$; f_α is a mapping of Φ_1^α to the point x_θ. The projections of the bicompacta T_1 and S_1 onto Γ (see item 1) will be denoted respectively by $\tilde{\omega}_T$ and $\tilde{\omega}_S$.

Proposition 1. If the set of points $x \in L_k$ for which $\operatorname{ind}_x L_k = 1$ is everywhere dense in L_k, $k = 0, 1$, then $\dim T_1 = \dim S_1 = 1$, $\operatorname{ind} T_1 = \operatorname{ind} S_1 = 2$. The set $\tilde{\omega}_S^{-1} Q_\Gamma^1 = \tilde{\omega}_S^{-1} T_0 \cap \tilde{\omega}_S^{-1} T_1$ has type G_δ in S_1, and $\operatorname{ind} \tilde{\omega}_S^{-1} \Gamma_k = 1$, $k = 0, 1$. If the bicompacta L_0 and L_1 possess the first axiom of countability, then the bicompactum S_1 will also possess the first axiom of countability.

Corollary. The bicompactum $S_1 = S_1(Q^1, Q^1, 1, 1)$ with the first axiom of countability can be represented as the sum of two subbicompacta S' and S'' of dimension $\operatorname{ind} S' = \operatorname{ind} S'' = 1$, whose intersection has type G_δ in S_1, but $\operatorname{ind} S_1 = 2$.

Remark 2. If the bicompacta L_0 and L_1 possess the first axiom of countability, then the bicompactum $S_1(L_0, L_1, a_0, a_1)$ is an irreducible image of a zero-dimensional bicompactum with the first axiom of countability under some mapping $\lambda = \lambda(L_0, L_1, a_0, a_1)$.

3. Bicompacta T^2 and S^2. By $R = \{r\}$ and $I = \{i\}$ we denote respectively the sets of rational and irrational points of the segment $Q^1 = \{t, 0 \le t \le 1\}$. Represent the set R as the disjoint sum of two sets $R_0 = \{r_0\}$ and $R_1 = \{r_1\}$ everywhere dense in Q^1. Consider the square $Q^2 = Q^1 \times Q^1$. We shall call marked the following pairs (x, F) of points $x \in Q^2$ and closed subsets $F \supset x$ in Q^2**: a) $x = (i', i'')$, $F = Q^2$; b) $x = (k, t)$, $t \in R_k \cup I$, $F = Q^2$, $k = 0, 1$; c) $x = (t^1, t^2)$, $0 < t^1 < 1$, $t^j \in$

* Everywhere in the note k is equal either to 0 or to 1, and if $k = 0$, then $\bar{k} = 1$, while if $k = 1$, then $\bar{k} = 0$.

** If the pair 1) $(x, +^{1/2}(x, j))$ or 2) $(x, +^{1/2}(x, 1) \cap +^{1/2}(x, 2))$ is marked, then in each of the cases 1) and 2) the pairs obtained from the indicated pair by replacing, in its second element, some (or all) signs $+$ by the sign $-$, are also considered marked.

$\in R$, $t_k \in I$, $j = 1, 2$, $j' = 2, 1$, $F = +^{1/2}(x, j)$; d) $x = (k, r_k)$, $F = +^{1/2}(x, 2)$, $k = 0, 1$; e) $x = (r^1, r^2)$, $0 < r^1 < 1$, $F = +^{1/2}(x, 1) \cap +^{1/2}(x, 2)$.

Denote the set of all marked pairs by B_1. Denote by B_2 the set of points x of the form (r^1, r^2), $0 < r^1 < 1$, $r^j \in R_0$, $r^{j'} \in R_1$, $j = 1, 2$, $j' = 2, 1$. Denote the sum $B_1 \cup B_2$ by B.

We construct the bicompactum T^2. Represent the set of all ordinal numbers $\alpha < \omega(\mathfrak{c})$ as the disjoint sum of sets $\mathfrak{A}_\theta$ of cardinality $\mathfrak{c}$, $\theta \in \Theta$, where the cardinality of Θ is also equal to $\mathfrak{c}$. Establish a one-to-one correspondence between the sets B and Θ. Here the subset of Θ corresponding to the set B_j will be denoted by $\Theta_j,\ j = 1, 2$. Let

$$\mathfrak{A}_j = \bigcup_{\theta\in\Theta_j} \mathfrak{A}_\theta,$$

$j = 1, 2$. To each $\alpha \in \mathfrak{A}_\theta,\ \theta \in \Theta_1$, assign a Cantor perfect set C_α. To each $\alpha \in \mathfrak{A}_\theta,\ \theta \in \Theta_2$, there corresponds a point $x_\theta = (r^1, r^2) \in B_2$. Put $a_0 = a_1 = x_\theta$ and $L_0^\theta = \{(r^1, t) : 0 \le t \le 1\},\ L_1^\theta = \{(t, r^2) : 0 \le t \le 1\}$. Denote by T_1^α the bicompactum $T_1(L_0^\theta, L_1^\theta, x_\theta, x_\theta)$, by $\tilde{\omega}_{T\alpha}$ the projection of T_1^α onto $L_0^\theta \cup L_1^\theta$, and by $\lambda_{T\alpha}$ an irreducible mapping of some zero-dimensional bicompactum T_0^α onto T_1^α. Put

$$\chi_3 = \bigcup_{\alpha\le\omega(\mathfrak{c})} \alpha \ \cup \bigcup_{\alpha\in\mathfrak{A}_\theta,\ \theta\in\Theta_1} C_\alpha \ \cup \bigcup_{\alpha\in\mathfrak{A}_2} T_0^\alpha.$$

The open sets in χ_3 are taken to be: the open subsets of the bicompacta $C_\alpha,\ \alpha \in \mathfrak{A}_1$, and $T_0^\alpha,\ \alpha \in \mathfrak{A}_2$; isolated numbers α; sets of the form

$$\bigcup_{\alpha'<\alpha\le\alpha''} \alpha \ \cup \bigcup_{\alpha'<\alpha<\alpha''} (C_\alpha \cup T_0^\alpha)$$

for limit numbers α''.

The bicompactum χ_3, obviously, is zero-dimensional. Finally, set T^2 equal to the bicompactum E from formula $(*)$, where $X = Q^2;\ \chi = \chi_3 \otimes$, and where: 1) for $\theta \in \Theta_1$ the set F_2^θ denotes the closed set $F = F_\theta$, while the set F_1^θ is the point x_θ from the marked pair (i.e., from the element of the set B_1) corresponding to the index θ; the bicompactum Φ_1^α is the square Q^2; $\Phi_0^\alpha = C_\alpha$; g_α coincides with some fixed mapping g of the Cantor perfect set C onto the square Q^2; f_α denotes the mapping of Q^2 to the point x_θ; 2) for $\theta \in \Theta_2$ we take $F_2^\theta = Q^2,\ F_1^\theta = L_0^\theta \cup L_1^\theta,\ \otimes\otimes\ \Phi_1^\alpha = T_1^\alpha,\ \Phi_0^\alpha = T_0^\alpha,\ f_\alpha = \tilde{\omega}_{T\alpha},\ g_\alpha = \lambda_{T\alpha}$.

We construct the bicompactum S^2. The product $P = Q_u^1 \times Q_\theta^1 \times D$ of the interval $Q_u^1 = \{u : 0 \le u \le 1\}$, the interval $Q_\theta^1 = \{\theta : 0 \le \theta \le 1\}$, and the pair of points $D = \{0, 1\}$ will be considered in the natural lexicographic ordering. Establish a one-to-one correspondence between the set Q_0^1 and the set B. Denote by Θ_j the subset of the set Q_θ^1 corresponding to the set $B_j,\ j = 1, 2$. Let further

$$\mathfrak{A}_\theta = \{\alpha = (u, \theta) : 0 \le u \le 1\}, \quad \theta \in Q_\theta^1; \qquad \mathfrak{A}_j = \bigcup_{\theta\in\Theta_j} \mathfrak{A}_{\theta,j} = 1, 2.$$

To each $\alpha \in \mathfrak{A}_1$ assign a Cantor perfect set C_α. Fix some mapping $g : C \to Q^2$.

To each $\alpha \in \mathfrak{A}_2$ there corresponds a point $x_\theta = (r^1, r^2) \in B_2$. Denote by S_1^α the bicompactum $S_1(L_0^\theta, L_1^\theta, x_\theta, x_\theta)$ (see the construction of T^2), by $\tilde{\omega}_{S\alpha}$ the projection of S_1^α onto $L_0^\theta \cup L_1^\theta$, and by $\lambda_{S\alpha}$ an irreducible mapping of some zero-dimensional bicompactum with the 1st axiom of countability S_0^α onto S_1^α $\boxtimes$. Put

$$\chi_4 = P \cup \bigcup_{\alpha\in\mathfrak{A}_1} C_\alpha \cup \bigcup_{\alpha\in\mathfrak{A}_2} S_0^\alpha.$$

The open sets in χ_4 are taken to be: the open subsets of the bicompacta $C_\alpha,\ \alpha \in \mathfrak{A}_1$, and $S_0^\alpha,\ \alpha \in \mathfrak{A}_2$; the points $(0,0,0)$ and $(1,1,1)$, sets of the form

$$((u_1,\theta_1,0),(u_2,\theta_2,1)) \cup \bigcup C_\alpha \cup \bigcup S_0^\alpha,$$

where the first term is an interval, and the summation in the second and third terms is taken over those $\alpha = (u,\theta)$ for which

$$(u_1,\theta_1,0) < (u,\theta,0) < (u,\theta,1) < (u_2,\theta_2,1).$$

Set S^2 equal to the bicompactum E from formula $(*)$, where $X = Q^2;\ \chi = \chi_4$ (further one must again read the part of the construction of the bicompactum T^2 from the sign $\otimes$ to the sign $\otimes\otimes$). $\Phi_1^\alpha = S_1^\alpha,\ \Phi_0^\alpha = S_0^\alpha,\ f_\alpha = \tilde{\omega}_{S\alpha},\ g_\alpha = \lambda_{S\alpha}$.

Theorem 1. The bicompactum S^2 has the 1st axiom of countability and

$$\dim S^2 = \dim T^2 = \operatorname{ind} S^2 = \operatorname{ind} T^2 = 2 < \operatorname{Ind} S^2 = \operatorname{Ind} T^2 = 3.$$

4. Bicompacta of PH. Denote by ζ the mapping of the Cantor perfect set $C = \{c\}$ onto the interval $Q^1 = \{v : 0 \le v \le 1\}$, identifying

corresponding to pairs of endpoints of intervals adjacent to C. The point c from C will be denoted, if necessary, by the corresponding point $\xi c = v$ from Q^1; moreover, if the point v corresponds to a pair of endpoints of some interval adjacent to C, then (when this is needed) the left endpoint of this interval will be denoted by v_l, and the right endpoint by v_r.

By χ_5 we denote the lexicographically ordered (with the interval topology) product of a countable collection of Cantor perfect sets $C_j = \{c\}$, $j = 1,2,\dots$. Choose on Q^1 a continuum of pairwise disjoint everywhere dense sets Q_α, $\alpha \in \mathfrak{A}$. For a given bicompactum $X = \{t\}$ of cardinality c we construct the bicompactum ΠX as follows. Let $h : X \to \mathfrak{A}$ denote a one-to-one correspondence between X and $\mathfrak{A}$. The elements of the partition ω of the product $\chi_5 \times X$ will be taken to be: (a) pairs of points

$$((c^1,\dots,c^{n-1},v_l^n,1,1,1,\dots),t)$$

and

$$((c^1,\dots,c^{n-1},v_r^n,0,0,0,\dots),t),$$

if there exists an index k, $1 \le k < n$, such that $\xi c^k \in Q_{h(t)}$, $t \in X$; (b) individual points not entering, for any t, into the pairs indicated in item (a). We denote the quotient space ω by ΠX.

If the natural projection of $\chi_5 \times X$ onto X is denoted by p, and the natural mapping of $\chi_5 \times X$ onto ΠX by ω, then there exists a mapping $\pi : \Pi X \to X$

satisfying the condition $p = \pi \cdot \omega$. We shall call the mapping π the projection of ΠX onto X.

Theorem 2. If X is a compactum and $\operatorname{ind} X = n$, then $\dim \Pi X = n$, $\operatorname{ind} \Pi X = n + 1$, $n = 1, 2, \ldots$. If, in addition, the compactum X is connected, then in ΠX there are two such nowhere dense disjoint closed sets F_1 and F_2 that, for any open set $O \subseteq [O] \subseteq X \setminus F_1$ or $\subseteq X \setminus F_2$, one always has $\operatorname{ind} \operatorname{fr} O \geq n - 1$. The bicompactum ΠX satisfies the first axiom of countability.*

5. The bicompactum T_1^2. Let $\Pi = \Pi Q^1$, where $Q^1 = \{t : 0 \leq t \leq 1\}$. By $R = \{r\}$ denote the set of points of χ_5 not having the form $(c^1, \ldots, c^n, 0, 0, 0, \ldots)$ or $(c^1, \ldots, c^n, 1, 1, 1, \ldots)$. Obviously, R is everywhere dense in χ_5. Represent R as the disjoint sum of the sets $R_0 = \{r_0\}$ and $R_1 = \{r_1\}$ dense in χ_5. By $0, 1$, and $I = \{i\}$ denote respectively the points $(0, 0, 0, \ldots)$ and $(1, 1, 1, \ldots)$ of χ_5, and the set $\chi_5 \setminus R$.

We note that on the segment Q^1 there also are points $0, 1$ and the sets R, R_0, R_1, and I (see item 3). Mark pairs (x, F)—points $x \in \Pi$ and closed sets $F \ni x$ in Π—the same as in subitems a)–e) of item 3, with Q^2 replaced by Π in subitems a) and b) (see Remark 1). The set of all marked pairs will be denoted by B_1. The set of points $x \in \Pi$ of the form (r^1, r^2), $0 < r^1 < 1$, $r^j \in R_0$, $r^{j'} \in R_1$, $j = 1, 2$; $j' = 2, 1$, will be denoted by B_2. The sum $B_1 \cup B_2$ will be denoted by B.

The further construction of T_1^2 is analogous to the construction of T_2, but instead of the bicompactum χ_3 one uses the "longer" bicompactum χ_6, constructed in the same way as χ_3, but with the use of ordinals $\alpha \leq \omega(2^c)$.**

Theorem 3. The inequalities

$$\dim T_1^2 = 1 < \operatorname{ind} T_1^2 = 2 < \operatorname{Ind} T_1^2$$

hold.

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Received
13 XI 1969

REFERENCES

1. B. A. Pasynkov, I. K. Lifanov, DAN, 192, No. 2 (1970).

2. V. V. Filippov, DAN, 192, No. 2 (1970).

3. I. K. Lifanov, V. V. Filippov, DAN, 192, No. 1 (1970).

* The existence of bicompacta satisfying the first axiom of countability with $\dim X = 1$, $\operatorname{ind} X > 1$ was established earlier, by another method, by V. Filippov (2).

** Initially, in the construction of T_1^2, instead of Π a bicompactum from (3) was used, to which I. K. Lifanov drew my attention.

Note: Figure translations are in progress. See original paper for figures.