CONDITIONS FOR UNIFORM RIESZ SUMMABILITY OF FOURIER SERIES WITH RESPECT TO AN ARBITRARY FUNDAMENTAL SYSTEM OF FUNCTIONS OF THE LAPLACE OPERATOR, FINAL IN THE CLASSES OF SOBOLEV, NIKOL' SKII, BESOV, LIOUVILLE, AND

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Abstract

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CONDITIONS FOR UNIFORM RIESZ SUMMABILITY OF FOURIER SERIES WITH RESPECT TO AN ARBITRARY FUNDAMENTAL SYSTEM OF FUNCTIONS OF THE LAPLACE OPERATOR, FINAL IN THE CLASSES OF SOBOLEV, NIKOL'SKII, BESOV, LIOUVILLE, AND ZYGMUND-HÖLDER

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This paper studies Fourier series with respect to the so-called fundamental systems of functions (f.s.f.) of the Laplace operator in an arbitrary subdomain Ω of an arbitrary N-dimensional domain G, i.e., Fourier series with respect to complete orthonormal in the domain G systems of functions $\{u_k(x)\}$, all elements $u_k(x)$ of which belong inside Ω to the class $C^{(2)}$ and, for certain nonnegative numbers λ_k, satisfy inside Ω the equations $\Delta u_k + \lambda_k u_k = 0$ (see (^{1})). At the same time, for the numbers λ_k (we shall call them fundamental numbers) finite points of condensation are allowed.

We study the Riesz means of the indicated Fourier series of any order s satisfying the inequalities $0 \leq s < (N-1)/2$, and for each of the classes of Sobolev, Nikol' skii, Besov, Liouville, and Zygmund-Hölder$*$ we establish final conditions ensuring uniform convergence and localization of the indicated Riesz means.

We pass to the exact formulation of the results. We shall consider functions $f(x)$ finite in the domain Ω and, in any case, belonging to the class $L_2(\Omega)$. Let f_k be the Fourier coefficients of the function $f(x)$ with respect to the system $\{u_k(x)\}$. Then the Riesz mean of order s of the function $f(x)$ will mean the sum

$$\sigma_\lambda^s(x) = \sum_{\lambda_k<\lambda} f_k u_k(x) \left(1 - \frac{\lambda_k}{\lambda}\right)^s. \tag{1}$$

For $s = 0$ the sum $\sigma_\lambda^s(x)$ becomes the partial sum

$$s_\lambda(x) = \sum_{\lambda_k<\lambda} f_k u_k(x)$$

of the Fourier series of the function $f(x)$.

We begin with the clarification of conditions which do not ensure even localization of the Riesz means (1) of any order s satisfying the conditions $0 \le s < (N-1)/2$.

Theorem 1 (on conditions that do not ensure localization of the Riesz means in the Zygmund-Hölder classes C^α**). *Let* $N \ge 2$*;* G *be an arbitrary* N*-dimensional domain;* $\{u_k(x)\}$ *be an arbitrary f.s.f. of the Laplace operator in any of its subdomains* Ω*;* x_0 *be any interior point of the domain* Ω*;* α *be any fixed number satisfying the inequalities* $0 < \alpha < (N-1)/2 - s$.

Then there exists a function $f(x)$ *satisfying the following conditions: 1)* $f(x)$ *is finite in the domain* Ω *and vanishes in some neigh*ness D of the point x_0; 2) $f(x) \in C^\alpha(\Omega)$; 3) the Riesz means (1) of the Fourier series of the function $f(x)$ have no limit at the point x_0 as $\lambda \to \infty$.

Corollary 1 (on conditions that do not ensure localization of Riesz means in the classes $H_p^\alpha, B_{p,\theta}^\alpha, L_p^\alpha, W_p^\alpha$). Let N, G, Ω, x_0, and α have the same meaning as in Theorem 1.

Then there exists a function $f(x)$ satisfying the following requirements: 1) $f(x)$ is finite in the domain Ω and vanishes in some neighborhood D of the point x_0; 2) $f(x)$ belongs in the domain Ω to each of the classes $H_p^\alpha, B_{p,\theta}^\alpha, L_p^\alpha, W_p^\alpha$ for arbitrary p and arbitrary θ; 3) the Riesz means (1) of the Fourier series of the function $f(x)$ have no limit at the point x_0 as $\lambda \to \infty$.

Thus, we have established that membership of a function $f(x)$ in any of the five classes listed above, with order of differentiability α less than $(N-1)/2 - s$, does not ensure even localization of the Riesz means $\sigma_\lambda^s(x)$ of the Fourier series of the function $f(x)$ (whatever the degree of summability p may be).

It is natural to ask about studying the Riesz means $\sigma_\lambda^s(x)$ for functions belonging to each of the indicated five classes with order of differentiability $\alpha \ge (N-1)/2 - s$. Since among the five classes indicated, $H_p^\alpha, B_{p,\theta}^\alpha, L_p^\alpha, W_p^\alpha$, and C^α, the Nikolskii class H_p^α is the broadest and contains all the other listed classes, it suffices to establish conditions ensuring Riesz summability of the Fourier series in terms of Nikolskii classes.

Theorem 2 (on conditions ensuring uniform Riesz summability in Nikolskii classes*). Let $N \ge 2$, let G be an arbitrary N-dimensional domain, $\{u_k(x)\}$ an arbitrary f.s.f. of the Laplace operator in any subdomain Ω of it; let $f(x)$

* Definitions of all classes used in this paper may be found in the monograph of S. M. Nikol' skii (^{2}).
** This theorem was proved by V. A. Il' in.

be an arbitrary function satisfying the following three requirements: 1) $f(x)$ is finite in the domain Ω; 2) $f(x)$ belongs in the domain Ω to the class H_2^α for $\alpha \geq (N-1)/2 - s$; 3) in some domain D contained in Ω, the function $f(x)$ belongs to the class B_p^α for $\alpha \geq (N-1)/2 - s,\ p\alpha > N$.

Then, uniformly with respect to x in every strictly interior subdomain D' of the domain D,

$$\lim_{\lambda\to\infty} \sigma_\lambda^s(x) = f(x).$$

Corollary 2. In the formulation of Theorem 2, instead of the Nikolskii class one may take any of the Besov, Liouville, Sobolev, or Zygmund-Hölder classes with the same order of differentiability α, the same degree of summability p, and (in the case of the Besov class) with arbitrary $\theta \geq 1$.

Comparing Theorem 2 and Corollary 2 with Theorem 1 and Corollary 1, we arrive at the conclusion that in each of the five classes under study we have established the final order of differentiability $\alpha \geq (N-1)/2 - s$ ensuring uniform Riesz summability of means of order s (for $\alpha < (N-1)/2 - s$, in each of the indicated classes even localization of Riesz means of order s will be absent). But the degree of summability p that we have found, satisfying the inequality $p\alpha > N$, is also final, since in any of the indicated classes the inequality $p\alpha \leq N$ admits the existence of an unbounded function whose Riesz means of the Fourier series certainly do not converge to it uniformly.

2. We now turn to the schemes of proof of Theorems 1 and 2. The proof of Theorem 1 is based on the following lemmas.

Lemma 1. Let G be an arbitrary N-dimensional domain; $\{u_k(x)\}$ an arbitrary f.s.f. of the Laplace operator in any subdomain Ω of it; x_0 any interior point of Ω.

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* This theorem was proved by Sh. A. Alimov.

Then for any $s \geq 0$ there exists a measurable set E, not containing the point x_0 and contained in Ω, such that for some $\beta > 0$ the inequality

$$\int_E \left| \sum_{\lambda_k<\lambda} u_k(x_0) u_k(y) \left(1 - \frac{\lambda_k}{\lambda}\right)^s \right| dy \geq \beta \lambda^{(N-1)/4 - s/2}$$

will hold.

Lemma 2. Let $s > 0$, $\beta > 0$, $s = r + \varkappa$, where r is an integer and $\varkappa$ satisfies the inequalities $0 < \varkappa \leq 1$; let $\sum u_k$ be any numerical series; the symbols s_λ, σ_λ^s, $\bar{s}_\lambda$, and $\bar{\sigma}_\lambda^s$ have the following meaning:

$$s_\lambda = \sum_{\lambda_k<\lambda} u_k, \qquad \sigma_\lambda^s = \sum_{\lambda_k<\lambda} u_k \left(1 - \frac{\lambda_k}{\lambda}\right)^s,$$

$$\bar{s}_\lambda = \sum_{\lambda_k<\lambda} u_k \lambda_k^\beta, \qquad \bar{\sigma}_\lambda^s = \sum_{\lambda_k<\lambda} u_k \lambda_k^\beta \left(1-\frac{\lambda_k}{\lambda}\right)^s.$$

Then the equality

$$\bar{\sigma}_\lambda^s = \lambda^\beta \sigma_\lambda^s + (-1)^{r+1} \frac{\beta}{\lambda^s} \int_0^\lambda \frac{d^{r+1}}{dt^{r+1}} \left[(\lambda-t)^s t^{\beta-1}\right] \frac{t^{r+1}}{(r+1)!}\, \sigma_t^{r+1}\, dt +$$

$$+(-1)^{r+1} \frac{s}{\lambda^s} \int_0^\lambda \frac{d^{r+1}}{dt^{r+1}} \left[(\lambda-t)^{s-1}(t^\beta - \lambda^\beta)\right] \frac{t^{r+1}}{(r+1)!}\, \sigma_t^{r+1}\, dt$$

holds.

From Lemmas 1 and 2 it follows that, under the conditions of Lemma 1, for any $0 \le s < (N-1)/2$ and any δ satisfying the condition $0 < \delta < (N-1)/4 - s/2$, the quantity

$$F_\lambda(x_0) = \int_E \left| \sum_{1\le\lambda_k<\lambda} \frac{u_k(x_0) u_k(y)}{\lambda_k^{(N-1)/4-s/2-\delta}} \left(1-\frac{\lambda_k}{\lambda}\right)^s \right| dy$$

is unbounded as $\lambda \to \infty$.

Further, for the proof of Theorem 1 a scheme is used that is very close to that set out on pp. 91-97 of the work (1).

For the proof of Theorem 2 the following three lemmas play an essential role.

Lemma 3. Let Ω_R be the subset of the domain Ω all points of which are at a distance from the boundary of Ω not less than the number $R > 0$, and let $\dot{H}_p^\alpha(\Omega_R)$ be the class of functions obtained as the closure in the metric H_p^α of the set of finite, infinitely differentiable functions in Ω_R. Suppose further that $f(x) \in \dot{H}_2^\alpha(\Omega_R)$ for $\alpha > 0$, $R > 0$. Then for any $\lambda > 0$

$$\sum_{\lambda<\lambda_n\le 2\lambda} f_n^2 \lambda_n^\alpha \le C_R \|f\|_{H_2^\alpha}^2.$$

Lemma 4. Let $f(x) \in H_p^\alpha(D)$, $p\alpha > N$, $\alpha = l + \varkappa$, where l is an integer, $0 < \varkappa \le 1$. Let further

$$\psi(r,x) = \frac{1}{\omega_N} \int_\theta f(x+r\theta)\, d\theta$$

be the mean value of the function f on the surface of the sphere of radius r with center at the point x; let $\varphi_m(r) = r^{m+\varkappa-1}\psi^{(m)}(r)$, where $m = 0, 1, \dots, l$. Then for any h from the interval $0 < h < R$, uniformly with respect to x in the subdomain D_{2R}, the estimates

$$\int_0^R |\varphi_m(r+h) - \varphi_m(r)|\, dr \le c\|f\|_{H_p^\alpha} h^\varkappa \qquad \text{for } 0 < \varkappa < 1,$$

$$\int_0^R |\varphi_m(r+2h) - 2\varphi_m(r+h) + \varphi_m(r)|\, dr \leq c\|f\|_{H_p^\alpha} h \qquad \text{for } \varkappa = 1.$$

Lemma 5. Let

$$F(r) = r^{N-1}\psi(r,x) = \frac{r^{N-1}}{\omega_N}\int_\theta' f(x+r\theta)\, d\theta,$$

let the symbol $V_\nu(t)$ denote the quantity $V_\nu(t) = \sqrt{t}J_\nu(t)$, where $J_\nu(t)$ is the Bessel function of order ν, and let the symbol DF denote the so-called Bessel differentiation

$$DF = \frac{d}{dr}\left[\frac{1}{r}F(r)\right],$$

with $D^{kF} = D(D^{k-1}F)$. Then, if $f(x) \in \dot{H}_2^\alpha(\Omega_R)$, $\alpha = l + \chi$, where l is an integer, $0 < \chi \leq 1$, then for $\nu = N/2 + s - l$, uniformly with respect to x in the subdomain Ω_{2R}, the estimate

$$\left|\sigma_\lambda^s(x) - 2^s\Gamma(s+1)2^{N/2}\Gamma(N/2+1)\chi^{\chi/2}\int_0^R V_\nu(r\sqrt{\lambda})\, r^{2l-N+\chi}D^{lF}\, dr\right| \leq$$

$$\leq C_R\|f\|_{\dot{H}_2^\alpha(\Omega_R)}.$$

From Lemmas 4 and 5 and from the obvious equality

$$r^{2l-N+\chi}D^{lF} = \sum_{m=0}^{l} A_{ml}\varphi_m^{(0)}(r),$$

valid with certain constants A_{ml}, the principal estimate follows:

$$|\sigma_\lambda^s(x)| \leq C_R\left[\|f\|_{\dot{H}_2^\alpha(\Omega_R)} + \|f\|_{H_p^\alpha(D)}\right], \tag{2}$$

valid uniformly with respect to x in the subdomain D_R, under the condition

$$f(x) \in \dot{H}_2^\alpha(\Omega_R), \qquad f(x) \in H_p^\alpha(D), \qquad \alpha \geq (N-1)/2 - s, \qquad p\alpha > N.$$

The principal estimate (2) completes the proof of Theorem 2.

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Note: Figure translations are in progress. See original paper for figures.

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