POSITIVE LINEAR FUNCTIONALS IN THE MINKOWSKI SENSE OVER CONVEX SURFACES

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Abstract

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POSITIVE LINEAR FUNCTIONALS IN THE MINKOWSKI SENSE OVER CONVEX SURFACES

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In the analysis of some extremal problems of isoperimetric type, as well as in a number of other questions, there arises the problem of representing a positive functional that is linear with respect to Minkowski operations over convex surfaces. In the present paper a description of such functionals is given in terms of an order relation related to the so-called strong ordering of measures in the sense of Loomis. However, unfortunately, the idea of the proof of a closely related theorem of Cartier–Fell–Meyer ([1]), describing the polar of the cone of convex functions, is not applicable in the present case. The subsequent exposition uses some properties of the space of convex sets.

Let $\mathfrak{B}_n$ be the collection of convex compact subsets of the n-dimensional arithmetic space R^n with the Euclidean norm $|\cdot|$. For $\mathfrak{x}, \mathfrak{y} \in \mathfrak{B}_n$ and $a \geq 0$, the Minkowski operations are defined by

$$\mathfrak{x} + \mathfrak{y} = \{z \in R^n : z = x + y \ (x \in \mathfrak{x};\ y \in \mathfrak{y})\};$$

$$a\mathfrak{x} = \{z \in R^n : z = ax \ (x \in \mathfrak{x})\}.$$

Endowing $\mathfrak{B}_n$ with the Hausdorff topology, we obtain a continuous semigroup with operators from the semigroup of nonnegative numbers R_+. Let now $\mathfrak{B}O_n$ be the set of solid convex compacta, or, what is the same, the set of convex surfaces. Since $\mathfrak{B}O_n$ is everywhere dense in $\mathfrak{B}_n$, the sought set $\mathfrak{B}O_n^*$ of positive linear functionals in the Minkowski sense coincides with $\mathfrak{B}_n^*$—the set of continuous R_+-operator homomorphisms of $\mathfrak{B}_n$ into R_+.

By $\mathrm{Sub}(R^n)$ we denote the collection of sublinear (convex, positively homogeneous) functions defined on all of R^n. We endow this set with the usual algebraic structure and with the topology of uniform convergence on compact subsets of R^n.

Denote by $\varphi : \mathfrak{B}_n \to \mathrm{Sub}(R^n)$ the mapping that sends a convex compactum $\mathfrak{x}$ to its support function $\varphi(\mathfrak{x})$, defined by the relation

$$\varphi(\mathfrak{x})(y) = \sup_{x \in \mathfrak{x}} (x, y) \qquad (y \in R^n).$$

The following is known.

Minkowski–Fenchel Theorem. *The mapping $\varphi : \mathfrak{B}_n \to \mathrm{Sub}(R^n)$ is an isomorphism of algebraic and topological structures.*

We shall now identify each sublinear function with its trace on the unit sphere $Z_n = \{x \in R^n : |x| = 1\}$. Under this identification, the elements of $\mathrm{Sub}(R^n)$ pass into the points of the cone H_n, defined by the relation

$$H_n = \left\{ h \in C(Z_n) : |x| h\left(\frac{x}{|x|}\right) + |y| h\left(\frac{y}{|y|}\right) - |x+y| h\left(\frac{x+y}{|x+y|}\right) \leq 0 \right.$$

$$\left. (x, y \in R^n) \right\}.$$

In the last formula, if $z = 0$, then, by definition, $|z| h(z/|z|) = 0$, and by $C(Z_n)$ is meant the space of functions continuous on Z_n with the Chebyshev norm.

From this point on, the symbol $\mathfrak{B}_n$ will be used to denote each of the three objects $\mathfrak{B}_n$, $\mathrm{Sub}(R^n)$, and $\bar{H}_n$.

It follows from the Stone–Weierstrass theorem that $\mathfrak{B}_n$ is total in $C(Z_n)$. Thus, the problem reduces to describing the dual cone

$$\mathfrak{B}_n^* = \{\mu \in C^*(Z_n) : \ \mu(h) \geq 0 \ (h \in \mathfrak{B}_n)\}.$$

Here $C^*(Z_n)$ is the space dual to $C(Z_n)$. In what follows we shall regard the identification of Radon measures with Borel measures on Z_n as already made.

Definition 1. Let $\mu, \nu \in C^*(Z_n)$. We shall say that μ is **linearly equivalent** to ν and write $\mu \sim \nu$, if $\mu(z) = \nu(z)$ for every linear function z from $\mathfrak{B}_n$.

Definition 2. For nonnegative measures $\mu, \nu \in C^*(Z_n)$ we shall say that μ is **linearly stronger** than ν, and write $\mu \gg \nu$, if for every finite decomposition $\nu_k \geq 0$,

$$\sum_{k=1}^{s} \nu_k = \nu$$

of the measure ν, there exists a decomposition $\mu_k \geq 0$,

$$\sum_{k=1}^{s} \mu_k = \mu$$

of the measure μ such that $\mu_k \sim \nu_k$ $(k = 1, 2, \dots, s)$.

It can be shown that the relation $\gg$ is a partial order. The principal result of the paper is the following.

Theorem 1. *The difference of nonnegative measures μ and ν belongs to $\mathfrak{B}_n^*$ if and only if μ is linearly stronger than ν.*

Sufficiency is checked directly on functions of the form

$$h = \sup_{1 \le k \le s} l_k,$$

where l_k is a linear functional on R^n. Moreover, the set of such functions is everywhere dense in $\mathfrak{B}_n$.

The proof of necessity is based on two lemmas.

Lemma 1. *Let μ, ν, δ be nonnegative measures, and suppose that the support of δ is finite. If $\mu + \delta \gg \nu + \delta$, then also $\mu \gg \nu$.*

Lemma 2. *Let $x_1, x_2, \dots, x_p, y_1, y_2, \dots, y_p$ be arbitrary vectors in R^n. If for every sublinear function $h \in \mathfrak{B}_n$ the inequality*

$$\sum_{k=1}^{p} h(x_k) \ge \sum_{k=1}^{p} h(y_k),$$

holds, then the vector $(y_1, y_2, \dots, y_p) \in (R^n)^p$ lies in the image of the set S of square stochastic matrices under the mapping

$$S \ni \|\alpha_k^s\| \mapsto \left(\sum_{k=1}^{p} \alpha_k^1 x_k,\ \sum_{k=1}^{p} \alpha_k^2 x_k,\ \dots,\ \sum_{k=1}^{p} \alpha_k^p x_k \right) \in (R^n)^p.$$

The proof of the theorem is completed as follows. It is clear that one must verify the implication $\mu - \nu \in \mathfrak{B}_n^* \Rightarrow \mu \gg \nu$. In view of Lemma 1, we may assume that the measure μ satisfies the hypotheses of the well-known theorem of A. A. Aleksandrov ([2]) on the reconstruction of a convex body from its surface function. Let $r \in \mathfrak{B}_n$ be such that $\mu = \mu(r)$. Here $\mu : \mathfrak{B}_n \to C^*(Z_n)$ is the mapping that sends a convex compactum to its surface function. Now let $\{r_m\}$ be a sequence of polytopes approximating r, and such that $2r \supset r_m \supset r$. As is known, in this situation $\{\mu(r_m)\}$ converges to $\mu(r)$ in the sense of the weak topology of the space $C^*(Z_n)$. Moreover, by the monotonicity of mixed volume, $\mu(r_m)(h) \ge \mu(r)(h)$ for every $h \in \mathfrak{B}_n$. Thus $\mu(r_m) - \nu \in \mathfrak{B}_n^*$, and consequently, by Lemma 2, $\mu(r_m) \gg \nu$. The required result is now established directly by passage to the limit.

As a consequence of the theorem proved, one obtains

Theorem 2. Let $\mathfrak{r}_1, \mathfrak{r}_2, \dots, \mathfrak{r}_{n-1}, \mathfrak{y}_1, \mathfrak{y}_2, \dots, \mathfrak{y}_{n-1}$ be convex surfaces from $\mathfrak{BO}_n$. Then the inequality

$$V(\mathfrak{r}_1, \mathfrak{r}_2, \dots, \mathfrak{r}_{n-1}, \mathfrak{z}) \geq V(\mathfrak{y}_1, \mathfrak{y}_2, \dots, \mathfrak{y}_{n-1}, \mathfrak{z})$$

is valid for every convex surface $\mathfrak{z}$ if and only if

$$\mu(\mathfrak{r}_1, \mathfrak{r}_2, \dots, \mathfrak{r}_{n-1}) \gg \mu(\mathfrak{y}_1, \mathfrak{y}_2, \dots, \mathfrak{y}_{n-1}).$$

Here $V(\cdot, \dots, \cdot)$ and $\mu(\cdot, \dots, \cdot)$ are, respectively, the mixed volume and the mixed surface function.

We now introduce into consideration the family of measures

$$NS = \{|x|\varepsilon_{x/|x|} + |y|\varepsilon_{y/|y|} - |x+y|\varepsilon_{(x+y)/|x+y|} \mid x, y \in \mathbb{R}^n,$$

where $\varepsilon_{z/|z|}$ for $z \neq 0$ is the measure generated by a unit mass located at the point $z/|z| \in Z_n$, and for $z = 0$ is the zero measure. The certainty that there can be no inequalities over sublinear functions other than "consequences" of the inequalities of sublinearity is confirmed by the following simple

Proposition. The closure in the weakened topology of the space $C^*(Z_n)$ of the conical convex hull $K(NS)$ of the set NS coincides with $\mathfrak{B}_n^*$.

For the proof it suffices to regard the spaces $C(Z_n)$ and $C^*(Z_n)$, with the weak and weakened topologies respectively, as a dual pair and to apply the separation theorem.

Remark. From the easily verified relation

$$K(NS) \subset \{\mu - \nu \in C^*(Z_n) : \ \mu \gg \nu\} \subset \mathfrak{B}_n^* \tag{1}$$

it is clear that, in order to prove the main result—theorem 1—it is enough to verify the weakened closedness of the middle set in (1). The author has not been able to carry out directly the indicated verification. We note that Cartie, Fell, and Meyer, in an analogous situation (see (1)), also did not follow this path.

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CITED LITERATURE

[1] R. Phelps, *Lectures on Choquet' s Theorems*, Moscow, 1968.
[2] H. Busemann, *Convex Surfaces*, "Nauka," 1964.

Note: Figure translations are in progress. See original paper for figures.