ON A METHOD FOR SOLVING THE CAUCHY PROBLEM FOR THE LINEARIZED BOLTZMANN EQUATION AND FOR COMPUTING SPACE-TIME CORRELATION FUNCTIONS

PHYSICS

1970

SovietRxiv

Abstract

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UDC 533.7

PHYSICS

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ON A METHOD FOR SOLVING THE CAUCHY PROBLEM FOR THE LINEARIZED BOLTZMANN EQUATION AND FOR COMPUTING SPACE-TIME CORRELATION FUNCTIONS

(Presented by Academician N. N. Bogolyubov, April 7, 1969)

The space-time correlation functions and, in particular, the following functions of a classical one-component gas (¹)

$$\begin{aligned} \widetilde{G}^{(00)}(\mathbf{r},t) &= \langle \hat{n}(0,0)\hat{n}(\mathbf{r},t)\rangle, & \widetilde{G}^{(01)}(\mathbf{r},t) &= \langle \hat{P}_\alpha(0,0)\hat{P}_\alpha(\mathbf{r},t)\rangle, \\ \widetilde{G}^{(02)}(\mathbf{r},t) &= \langle \hat{\Pi}_{\alpha\beta}(0,0)\hat{\Pi}_{\alpha\beta}(\mathbf{r},t)\rangle, & \widetilde{G}^{(11)}(\mathbf{r},t) &= \langle \hat{S}_\alpha(0,0)\hat{S}_\alpha(\mathbf{r},t)\rangle \end{aligned} \tag{1}$$

in the Boltzmann limit can be represented in the form

$$G^{(\mu\nu)}(\mathbf{r},t) = (\psi_{\mu\nu}, \tilde{g}_{\mu\nu}), \tag{2}$$

where the functions $\tilde{g}_{\mu\nu}(\mathbf{r},\xi,t)$ satisfy the linearized Boltzmann equation

$$\partial\tilde{g}/\partial t + \xi\,\partial\tilde{g}/\partial\mathbf{r} = J(\tilde{g}) \tag{3}$$

and the initial conditions

$$\tilde{g}_{\mu\nu}(t=0) = \psi_{\mu\nu}\delta(\mathbf{r}), \tag{4}$$

while $\psi_{\mu\nu}(\xi_x, \xi^2)$ is an orthonormal system of Burnett polynomials (²), complete in the Hilbert space $L^2(\omega)$ of functions $\varphi(\xi_x, \xi^2)$ with scalar product

$$(\varphi_1, \varphi_2) = \int \omega\varphi_1\overline{\varphi}_2\, d^3\xi, \qquad \omega(\xi) = (2\pi)^{-3/2}\exp(-\xi^2/2). \tag{5}$$

In relations (1), dimensionless dynamical variables are introduced which depend on dimensionless coordinates, momenta, and time (the reduction of the latter to dimensionless form is standard; cf. (^{2})), the numerical factors being chosen so that at $t = 0$ the functions $\widetilde{G}_{\mu\nu}$ become δ-functions (the functions $\hat{P}_\alpha$, $\hat{\Pi}_{\alpha\beta}$, $\hat{S}_\alpha$ are multiplied by $(1/3)^{1/2}$, $(1/10)^{1/2}$, $(5/6)^{1/2}$, respectively). In (1), $\hat{n}(\mathbf{r})$, $\hat{P}_\alpha(\mathbf{r})$, $\hat{\Pi}_{\alpha\beta}(\mathbf{r})$, $\hat{S}_\alpha(\mathbf{r})$ are the dynamical variables of the number density of particles, momentum, the traceless part of the momentum-flux tensor, and the energy flux (these notations correspond to those in (^{1})).

Passing to the Fourier representation in the spatial variables, we reduce the Cauchy problem (3), (4) to the form

$$\partial g/\partial t = J(g) - ik\xi_x g, \qquad g_{\mu\nu}(k, \xi, t = 0) = \psi_{\mu\nu}, \tag{6}$$

where the transformed functions $G^{(\mu\nu)}(k, t)$ and $g_{\mu\nu}(k, \xi, t)$ are related to one another by the same relations (2), if the tilde is omitted in them. The dimensionless wave number k is, in order of magnitude, equal to the ratio of the mean free path to the wavelength of the Fourier component under consideration, i.e., to the Knudsen number.

We shall consider the linearized collision operator

$$J(g) = \int \omega(\xi_1)\,[g(\xi_1') + g(\xi') - g(\xi_1) - g(\xi)]\,B(\theta, U)\,d\varepsilon\,d\theta\,d^3\xi_1 \tag{7}$$

as an operator acting in the Hilbert space $L^2(\omega)$. The domain $D(J)$ of the operator J is dense in $L^2(\omega)$, and the operator J is symmetric and nonpositive, i.e.,

$$(g, Jf) = (Jg, f), \qquad (Jg, g) \leqslant 0, \qquad g, f \in \mathcal{D}(J), \tag{8}$$

where from the equality $(Jg, g) = 0$ it follows that g is a linear combination of the functions $\psi_{00}, \psi_{01}, \psi_{10}$ (Boltzmann' s H-theorem), i.e., zero is a threefold degenerate eigenvalue of the operator J.

For molecules described by a power-law interaction potential corresponding to repulsive forces, with exponent greater than 4, and with finite range of action, or by a potential with a hard core and finite range of action, the operator J has a continuous spectrum contained in the interval $(-\infty, -\nu_0)$, $\nu_0 > 0$ ($^{4-6}$). In addition, for potentials with a hard core, in the interval $(-\nu_0, 0)$ there are eigenvalues of finite multiplicity (^{7}), a large number of which have been found numerically (^{8}).

In the case of Maxwellian molecules the cross section $B(\theta, U)$ does not depend on U, and the operator J has a discrete spectrum of eigenvalues $\lambda_{\mu\nu}$, to which there corresponds a complete system in $L^2(\omega)$ of eigenfunctions $\psi_{\mu\nu}(\xi)$.

Using the property of semiboundedness of the operator $\mathcal{L} = \mathcal{L}(\varepsilon) = J + \varepsilon\xi_x$, $\varepsilon = -ik$:

$$\operatorname{Re}(\mathcal{L}f, f) \leqslant 0, \qquad f \in \mathcal{D}(\mathcal{L}), \tag{9}$$

one can prove the theorem of existence and uniqueness of the solution of the following Cauchy problem for the linearized Boltzmann equation:

$$\partial g/\partial t = \mathcal{L}g + \varphi_1, \qquad g(t=0) = \varphi_2. \tag{10}$$

Theorem (3). Let $\varphi_2 \in \mathcal{D}(\mathcal{L})$ and let the function $\varphi_1(t) \in L^2(\omega)$ be continuously differentiable with respect to t. Then in $L^2(\omega)$ there exists a unique function $g(t)$ possessing the following properties:
a) $g(t) \in \mathcal{D}(\mathcal{L})$; $g(t)$ is continuously differentiable with respect to t;
b) $(d/dt)g(t) = \mathcal{L}g(t) + \varphi_1(t)$;
c) $\lim\limits_{t\to 0} \|g(t) - \varphi_2\| = 0$.

Suppose that the spectrum and the orthonormal eigenfunctions of the operator $\mathcal{L}$ are known, i.e., the solutions of the equation

$$\mathcal{L}(\varepsilon)\psi_{\mu\nu}(\varepsilon) = \lambda_{\mu\nu}(\varepsilon)\psi_{\mu\nu}(\varepsilon), \tag{11}$$

forming a complete system in the space $L^2(\omega)$, are known.

Then the solution of problem (10) has the form

$$g = \sum_{(\mu\nu)} \alpha_{\mu\nu}(t) \exp[\lambda_{\mu\nu}(\varepsilon)t]\psi_{\mu\nu}(\varepsilon),$$

$$\alpha_{\mu\nu}(t) = (\varphi_2, \varphi_{\mu\nu}(\varepsilon)) + \int_0^t e^{-\lambda_{\mu\nu}(\varepsilon)\tau}(\varphi_1(\tau), \varphi_{\mu\nu}(\varepsilon))\, d\tau, \tag{12}$$

where $\varphi_{\mu\nu}(\varepsilon) = \overline{\psi_{\mu\nu}(\varepsilon)}$ is an eigenfunction of the adjoint operator $\mathcal{L}^*(\varepsilon) = \overline{\mathcal{L}(\varepsilon)}$, belonging to the eigenvalue $\overline{\lambda_{\mu\nu}(\varepsilon)}$.

We shall carry out the further consideration for Maxwellian molecules and for the case when the functions φ_1 and φ_2 are finite linear combinations of eigenfunctions $\psi_{\mu\nu}$ of the operator J. To construct an approximate solution we use the smallness of the parameter k and apply perturbation theory to determine the solutions of equation (11). Although the conditions for applicability of perturbation theory (9) in the case of Maxwellian molecules do not

are satisfied, it will be shown that the constructed solution is asymptotic as $k \to 0$. Substituting the expansions

$$\lambda_{\mu\nu}(\varepsilon) = \lambda_{\mu\nu} + \sum_{k=1}^{\infty} \varepsilon^k \lambda_{\mu\nu}^{(k)}, \qquad \psi_{\mu\nu}(\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \psi_{\mu\nu}^{(k)}, \tag{13}$$

into (11), for determining the corrections $\psi_{\mu\nu}^{(k)}$ and $\lambda_{\mu\nu}^{(k)}$ we have the equations

$$(J-\lambda_{\mu\nu})\psi_{\mu\nu}^{(k)} = -\xi_x\psi_{\mu\nu}^{(k-1)} + \sum_{m=0}^{k-1}\lambda_{\mu\nu}^{(k-m)}\psi_{\mu\nu}^{(m)}, \qquad \lambda_{\mu\nu}^{(k)} = (\psi_{\mu\nu}^{(0)}, \xi_x\psi_{\mu\nu}^{(k-1)}). \quad (14)$$

We represent the solution of the Cauchy problem (10) in the form

$$g = g_1 + g^{(n)}, \quad (15)$$

where the function $g^{(n)}$ is determined by formulas (12), in which, instead of the numbers $\lambda_{\mu\nu}(\varepsilon)$ and the functions $\psi_{\mu\nu}(\varepsilon)$, the expressions

$$\lambda_{\mu\nu}^{(n)}(\varepsilon) = \sum_{k=0}^{n}\varepsilon^k\lambda_{\mu\nu}^{(k)}, \qquad \Psi_{\mu\nu}^{(n)}(\varepsilon) = \frac{\psi_{\mu\nu}^{(n)}(\varepsilon)}{\|\psi_{\mu\nu}^{(n)}(\varepsilon)\|}, \qquad \psi_{\mu\nu}^{(n)}(\varepsilon) = \sum_{k=0}^{n}\varepsilon^k\psi_{\mu\nu}^{(k)} \quad (16)$$

are taken.

The function g_1 is the solution of the Cauchy problem

$$\frac{\partial g_1}{\partial t} = \mathcal{L}g_1 + h_1, \qquad g_1(t=0) = h_2 = \varphi_2 - \sum_{(\mu\nu)}(\varphi_2, \Phi_{\mu\nu}^{(n)}(\varepsilon))\Psi_{\mu\nu}^{(n)}(\varepsilon); \quad (17)$$

$$h_1 = \varphi_1 - \sum_{(\mu\nu)}(\varphi_1, \Phi_{\mu\nu}^{(n)}(\varepsilon))\Psi_{\mu\nu}^{(n)}(\varepsilon) + \sum_{(\mu\nu)}\alpha_{\mu\nu}^{(n)}(t)\exp[\lambda_{\mu\nu}^{(n)}(\varepsilon)t]\times$$

$$\times[\mathcal{L} - \lambda_{\mu\nu}^{(n)}(\varepsilon)]\Psi_{\mu\nu}^{(n)}(\varepsilon). \quad (18)$$

Using expressions (14), we find

$$(\mathcal{L} - \lambda_{\mu\nu}^{(n)}(\varepsilon))\psi_{\mu\nu}^{(n)}(\varepsilon) = \varepsilon^{n+1}\xi_x\psi_{\mu\nu}^{(n)} + \sum_{p=1}^{n}\varepsilon^{n+p}\sum_{m=0}^{n-p}\lambda_{\mu\nu}^{(p+m)}\psi_{\mu\nu}^{(n-m)}, \quad (19)$$

whence it follows that

$$\left|(\Psi_{\mu\nu}^{(n)}(\varepsilon), \Psi_{\mu'\nu'}^{(n)}(\varepsilon))\right| = \delta_{\mu\mu'}\delta_{\nu\nu'} + O(|\varepsilon|^{n+1}). \quad (20)$$

Since the function $\xi_x\psi_{\mu\nu}$ is a finite linear combination of the functions $\psi_{\mu\nu}$, the functions $\psi_{\mu\nu}^{(k)}$ also, by construction, possess this property. Therefore, from (19) and (20) it follows that $\|h_1\| \le C_1|\varepsilon|^{n+1}$, $\|h_2\| \le C_2|\varepsilon|^{n+1}$, whence, by virtue of the existence and uniqueness theorem, it follows that $\|g_1\| \le M|\varepsilon|^{n+1}$, i.e. the following condition of asymptotic convergence of the method for solving the Cauchy problem by means of perturbation theory:

$$\lim_{k\to 0} \frac{1}{k^n}\|g - g^{(n)}\| = 0. \tag{21}$$

We note that, for all computed corrections to the eigenvalues, the inequality $(-1)^n \lambda_{\mu\nu}^{(2n)} < 0$ holds, i.e. one may assume $\operatorname{Re}\lambda_{\mu\nu}^{(n)}(-ik) < 0$ (cf. also (^{2})). The numbers $\lambda_{\mu\nu}^{(n)}(\varepsilon)$ are the roots of the dispersion equation for the linearized Boltzmann equation. Formula (12) shows how these roots appear in the solution of the Cauchy problem for the linearized Boltzmann equation. Apparently, this property is a direct consequence of the semiboundedness of the operator $\mathcal{L}$; however, its proof in the general case is unknown. Then relation (21) is valid uniformly in t, $t \in [0, \infty)$.

In the case of hard spherical molecules, in addition to the discrete spectrum, there is a continuous spectrum of the operator J; however, if the system of eigenfunctions

functions belonging to the discrete spectrum is complete in $L^2(\omega)$, the solution of the Cauchy problem can also be constructed using perturbation theory, all the more so because in this case the perturbation-theory series determining the eigenvalues and eigenfunctions converge ^{10}.

Using the approximate eigenfunctions and numbers (16), we shall write the expressions for the correlation functions in the form

$$G^{(\mu\nu)}(k,t) = \sum_{(pl)} \left|(\psi_{\mu\nu}, \Psi_{pl}^{(n)}(\varepsilon))\right|^2 \exp[\lambda_{pl}^{(n)}(\varepsilon)t]. \tag{22}$$

For finite n, the series (22) is finite, since the function $\psi_{\mu\nu}$ enters only a finite number of functions $\Psi_{pl}^{(n)}(\varepsilon)$. All terms of the series (22) decrease exponentially with time, which confirms the usually accepted hypothesis of exponential decay of correlation functions.

The discrete spectrum of the linearized collision operator for Maxwellian molecules consists of simple, doubly degenerate ($\lambda_{\mu 0} = \lambda_{\mu-1,1}$) and two triply degenerate ($\lambda_{00} = \lambda_{01} = \lambda_{10} = 0$, $\lambda_{02} = \lambda_{30} = \lambda_{21}$) eigenvalues. Accordingly, the numbers $\lambda_{\mu\nu}(-ik) = r_{\mu\nu}(k) + ij_{\mu\nu}(k)$ can be divided into real and complex ones. The real numbers are $\lambda_{\mu\nu}(-ik)$ with $\nu \ne 0, 1$ (this also includes $\lambda_{02}(-ik)$) and $\lambda_{00}(-ik)$. All the remaining numbers are complex, and moreover $\lambda_{\mu 0}(-ik) = \overline{\lambda_{\mu-1,1}(-ik)}$. In addition, it can be shown that

$$\left|(\psi_{\mu\nu}, \Psi_{p0}^{(n)}(-ik))\right|^2 = \left|(\psi_{\mu\nu}, \Psi_{p-1,1}^{(n)}(-ik))\right|^2,$$

so that expression (22) can be represented in the form

$$G^{(\mu\nu)}(k,t) = \sum_{\substack{(pl)\\(l\neq 0,1)}} \left|(\psi_{\mu\nu}, \Psi^{(n)}_{pl}(-ik))\right|^2 \exp[r^{(n)}_{pl}(k)t]+$$

$$+2\sum_{p=1}^{\infty} \left|(\psi_{\mu\nu}, \Psi^{(n)}_{p0}(-ik))\right|^2 \exp[r^{(n)}_{p0}(k)t]\cos j_{p0}(k)t. \tag{23}$$

Making also a Laplace transform in time, from (22) we obtain

$$G^{(\mu\nu)}(k,z) = \sum_{(pl)} \frac{\left|(\psi_{\mu\nu}, \Psi^{(n)}_{pl}(-ik))\right|^2}{z-\lambda^{(n)}_{pl}(-ik)}. \tag{24}$$

From (24) it is easy to obtain an expression for the spectral function of the intensity of scattered light $S(k,\omega)$. For this one should put $z = i\omega$, $(\mu\nu) = (00)$ in (24) and take the real part of the resulting function.

The author expresses gratitude to D. N. Zubarev, V. B. Lidskii, and B. V. Paltsev for valuable discussion.

Received
31 III 1969

CITED LITERATURE

[1] A. D. Khon' kin, DAN, **183**, 1285 (1968).
[2] L. Sirovich, Phys. Fluids, **6**, 10 (1963).
[3] G. Scharf, Helv. phys. acta, **40**, 929 (1967).
[4] H. Grad, Phys. Fluids, **6**, 147 (1963).
[5] A. A. Arsen' ev, DAN, **165**, 1104 (1965).
[6] J. R. Dorfman, Proc. Nat. Acad. Sci., **50**, 804 (1963).
[7] G. W. Ford, M. Schreiber, Proc. Nat. Acad. Sci. U.S.A., **60**, 802 (1968).
[8] C. L. Pekeris et al., Phys. Fluids, **5**, 1608 (1962).
[9] F. Riesz, B. Sz.-Nagy, *Lectures on Functional Analysis*, IL, 1954.
[10] A. A. Arsen' ev, Zhurn. vychisl. matem. i matem. fiz., **6**, 375 (1966).

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