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Abstract

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Mathematics

E. Mukhamadiev

On the Computation of the Rotation of Finite-Dimensional Vector Fields

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Many applications of topological methods to the study of various problems in the qualitative theory of differential equations and in the theory of nonlinear operator equations require the computation or estimation of the rotation (see ([1,2])) of vector fields. For broad classes of vector fields, the computation of the rotation reduces to the computation of the degree of sufficiently simple mappings (see ([1–4])). However, in a number of cases the computation of the rotation requires overcoming specific difficulties (we mention here the papers ([5–8])).

In the present article, theorems are proposed that make it possible to estimate the rotations of a vector field in new cases.

1. Let G be a bounded domain in the real $2n$-dimensional Euclidean space R^{2n} with boundary Γ. We shall denote the points of this space by

$$x = \{x_1, x_2, \dots, x_{2n-1}, x_{2n}\}. \tag{1}$$

Let a vector field be given on G,

$$\Phi x = \{f_1(x), \dots, f_{2n}(x)\}, \tag{2}$$

whose components $f_i(x)$ are continuously differentiable functions. Then the field (2) is differentiable on G in the sense that the increment $\Phi(x+h) - \Phi x$ $(x, x+h \in G)$ can be represented in the form

$$\Phi(x+h) - \Phi x = \Phi'(x)h + o(\|h\|),$$

where $\Phi'(x)$ is a linear operator in R^{2n}, or, equivalently, $\Phi'(x)$ is a square matrix of order $2n$.

If the field (2) has no zero vectors on the boundary Γ of the domain G, then the rotation $\gamma(\Phi;\Gamma)$ of this field on Γ is defined. The rotation $\gamma(\Phi;\Gamma)$ is the degree of the mapping

$$Fx = \Phi x / \|\Phi x\| \quad (x \in \Gamma)$$

of the boundary Γ onto the unit sphere.

Theorem 1. *Suppose that the field (2) has no zero vectors on* Γ*. Suppose that for every fixed* $x \in G$ *there can be specified a square matrix* $T(x)$ *of order* $2n$*, without real eigenvalues, such that*

$$\Phi'(x)T(x) = T(x)\Phi'(x) \quad (x \in G). \tag{3}$$

Then

$$\gamma(\Phi;\Gamma) \geq 0.$$

This theorem includes as a special case the known assertion (see ([8])) on the nonnegativity of the degree of analytic mappings in complex spaces. Indeed, if R^{2n} coincides with the n-dimensional complex space C^n and if the coordinates in (1) are numbered so that the point (1) in the space R^{2n} corresponds to the point $\{x_1+$

$+ix_2,\ x_3 + ix_4, \dots, x_{2n-1} + ix_{2n}\}$ in the space C^n, then for analytic mappings Φ in C^n the conditions (3) are fulfilled for the constant matrix T, where

$$Tx = \{x_1 - x_2,\ x_1 + x_2, \dots, x_{2n-1} - x_{2n},\ x_{2n-1} + x_{2n}\}.$$

Equality (3) in this case coincides with the Cauchy–Riemann condition for functions of many complex variables (9).

Let the field (2) be generated by a continuously differentiable mapping of a k-dimensional space of quaternions into itself. This space is, obviously, a $4k$-dimensional real space. Then the field Φ satisfies condition (3) with the constant matrix

$$T = \begin{pmatrix} T_1 & 0 & \dots & 0 \\ 0 & T_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & T_k \end{pmatrix},$$

where

$$T_i = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

As is not difficult to see, the matrix T has no real eigenvalues. Therefore, from Theorem 1 it follows that

Theorem 2. *Let a differentiable vector field Φ in quaternionic space E not vanish on the boundary Γ of a bounded domain. Then $\gamma(\Phi;\Gamma) \geqslant 0$.*

2. **Theorem 3.** *Let on the boundary Γ of a domain $G \subset R^{2n}$ the field Φ have no zero vectors. Suppose that for each fixed $x \in G$ there can be specified a square matrix $T(x)$ of order $2n$, without real eigenvalues, such that*

$$\Phi'(x)T(x) = -T(x)\Phi'(x) \qquad (x \in G).$$

Then

$$(-1)^n \gamma(\Phi;\Gamma) \geqslant 0.$$

3. A continuous mapping U of the sphere S^n ($\|x\| = 1,\ x \in R^{n+1}$) into itself is called periodic with period p if $U^i \neq I,\ i = 1, \dots, p-1$, and $U^p = I$. A periodic mapping is a homeomorphism of the sphere S^n onto itself. Along with the mapping U on S^n, consider a periodic mapping V of period q, where q is a divisor of p. By γ_U and γ_V we denote respectively the degrees of the mappings U and V of the sphere S^n onto itself.

Theorem 4. *Let U and V be simplicial mappings with respect to certain subdivisions of the sphere S^n. Suppose $U, U^2, \dots, U^{p-1}$ have no fixed points on S^n. Suppose*

$$\gamma_U \cdot \gamma_V = 1.$$

Let continuous vector fields Φ and Ψ, without zero vectors on S^n, satisfy the conditions

$$\frac{\Phi U^i x}{\|\Phi U^i x\|} \neq -V^i \left(\frac{\Phi x}{\|\Phi x\|} \right) \qquad (i = 1, \dots, p-1); \tag{4}$$

$$\frac{\Psi U^i x}{\|\Psi U^i x\|} \neq -V^i \left(\frac{\Psi x}{\|\Psi x\|} \right) \qquad (i = 1, \dots, p-1). \tag{5}$$

Then

$$\gamma(\Phi; S^n) = \gamma(\Psi; S^n) \pmod{p}.$$

If the mapping V has a fixed point x_0 on S^n, then condition (5) is satisfied by the vector field $\Psi x \equiv x_0$. Consequently, in this case it follows from (4) that $\gamma(\Phi; S^n) \equiv 0(\bmod\, p)$.

Theorem 5. Let U and V be mappings of the sphere S^n into itself (not necessarily periodic). Let $\gamma_U \cdot \gamma_V = -1$. Finally, suppose that the continuous vector field Φ, without zero vectors on S^n, satisfies the condition

$$\frac{\Phi U x}{\|\Phi U x\|} \neq -V\left(\frac{\Phi x}{\|\Phi x\|}\right).$$

Then

$$\gamma(\Phi; S^n) = 0.$$

Theorems 4 and 5 supplement and develop the results of M. A. Krasnosel' skii presented in [6]. The proof of Theorem 4 uses these results from [6].

4. In conclusion we consider a vector field Φ in the two-dimensional plane. Let U and V be mappings of the circle S^1 into itself of periods p and q, where $p = dq$, such that $U, U^2, \dots, U^{p-1}, V, V^2, \dots, V^{q-1}$ have no fixed points on S^1.

Theorem 6. Suppose that the continuous vector field Φ on S^1 has no zero vectors and satisfies the condition

$$\frac{\Phi U^i x}{\|\Phi U^i x\|} \neq -V^i\left(\frac{\Phi x}{\|\Phi x\|}\right) \qquad (x \in S^1,\ i = 1, \dots, p-1).$$

Then

$$\gamma(\Phi; S^1) \equiv d(\bmod\, p).$$

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Voronezh
State University

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CITED LITERATURE

[1] P. S. Aleksandrov, *Combinatorial Topology*, Moscow, 1947.
[2] M. A. Krasnosel' skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Moscow, 1956.
[3] S. Lefschetz, *Algebraic Topology*, Moscow, 1949.
[4] P. Hilton and S. Wylie, *Homology Theory*, Moscow, 1966.
[5] P. Smith, *Ann. J. Math.*, (2), 42 (1941).
[6] M. A. Krasnosel' skii, *DAN*, 101, No. 3 (1955).
[7] T. P. Zabreiko and M. A. Krasnosel' skii, *Sibirsk. Mat. Zhurn.*, 5, No. 3 (1964).
[8] J. Cronin, *Fixed Points and Topological Degree in Nonlinear Analysis*, Providence, 1964.
[9] V. S. Vladimirov, *Methods of the Theory of Functions of Several Complex Variables*, Moscow, 1964.

Note: Figure translations are in progress. See original paper for figures.