ASYMPTOTICS OF THE SOLUTION OF A PROBLEM WITH AN INITIAL JUMP FOR SECOND-ORDER HYPERBOLIC EQUATIONS CONTAINING A SMALL PARAMETER

MATHEMATICS

1970

SovietRxiv

Abstract

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UDC 517.946

MATHEMATICS

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ASYMPTOTICS OF THE SOLUTION OF A PROBLEM WITH AN INITIAL JUMP FOR SECOND-ORDER HYPERBOLIC EQUATIONS CONTAINING A SMALL PARAMETER

(Presented by Academician A. N. Tikhonov on April 9, 1970)

1. Consider the Cauchy problem for hyperbolic equations of the form:

$$\varepsilon\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}\right)=A(t,x,u)\frac{\partial u}{\partial t}+B(t,x,u), \tag{1}$$

$$u(t,x)\big|_{t=0}=\varphi(x), \qquad \varepsilon\frac{\partial u}{\partial t}\bigg|_{t=0}=\psi(x), \tag{2}$$

where $\varepsilon>0$ is a small parameter. Suppose that in the strip $(t\geq 0,\ -\infty<x<+\infty)$, for all u,

$$A(t,x,u)\geq\gamma>0. \tag{3}$$

In the present paper it is proved that the solution u of problem (1), (2), as $\varepsilon\to 0$, will tend to the solution of the reduced equation

$$0=A(t,x,u_0)\frac{\partial u_0}{\partial t}+B(t,x,u_0), \tag{4}$$

but, however, the solution of equation (4) does not satisfy the former initial condition (2), and satisfies an entirely different condition:

$$u_0(t,x)\big|_{t=0}=\varphi(x)+\Delta(x), \tag{5}$$

where the function $\Delta(x)$ will be called the initial jump of the function u, and it is determined uniquely from the following equation:

$$\psi(x) = \int_{\varphi(x)}^{\varphi(x)+\Delta(x)} A(0,x,u)\,du. \tag{6}$$

We now turn to the question of constructing the asymptotics, with respect to the small parameter ε, of the solution of problem (1), (2).

2. We shall seek an approximate solution of problem (1), (2) in the form of an expansion in integral powers of the small parameter ε

$$u(t,x,\varepsilon) = u_0(t,x) = \varepsilon u_1(t,x) + \dots + w_0(\tau,x) + \varepsilon w_1(\tau,x) + \dots, \tag{7}$$

where $\tau = t/\varepsilon$. Substituting expansion (7) into equation (1), we obtain:

$$\begin{aligned}
&\varepsilon^2\left(\frac{\partial^2 u_0(t,x)}{\partial x^2} + \varepsilon\frac{\partial^2 u_1(t,x)}{\partial x^2} + \dots + \frac{\partial^2 w_0(\tau,x)}{\partial x^2} + \varepsilon\frac{\partial^2 w_1(\tau,x)}{\partial x^2} + \dots\right)\\
&\quad -\varepsilon^2\left(\frac{\partial^2 u_0(t,x)}{\partial t^2} + \varepsilon\frac{\partial^2 u_1(t,x)}{\partial t^2} + \dots\right) - \left(\frac{\partial^2 w_0(\tau,x)}{\partial \tau^2} + \varepsilon\frac{\partial^2 w_1(\tau,x)}{\partial \tau^2} + \dots\right)\\
&= \Big[A(\varepsilon\tau, x, u_0(\varepsilon\tau,x) + \varepsilon u_1(\varepsilon\tau,x) + \dots + w_0(\tau,x) + \varepsilon w_1(\tau,x) + \dots)\frac{\partial}{\partial\tau}(u_0(\varepsilon\tau,x)\\
&\quad +\varepsilon u_1(\varepsilon\tau,x) + \dots + w_0(\tau,x) + \varepsilon w_1(\tau,x) + \dots)\\
&\quad -A(\varepsilon\tau, x, u_0(\varepsilon\tau,x) + \varepsilon u_1(\varepsilon\tau,x) + \dots)\frac{\partial}{\partial\tau}(u_0(\varepsilon\tau,x) + \varepsilon u_1(\varepsilon\tau,x) + \dots)\Big]\\
&\quad + \varepsilon A(t,x,u_0(t,x)+
\end{aligned}$$

$$\begin{aligned}
&+\varepsilon u_1(t,x) + \cdots)\frac{\partial}{\partial t}(u_0(t,x) + \varepsilon u_1(t,x) + \cdots) + \varepsilon\big[B(\varepsilon\tau, x, u_0(\varepsilon\tau,x)+\\
&\quad +\varepsilon u_1(\varepsilon\tau,x) + \cdots + w_0(\tau,x) + \varepsilon w_1(\tau,x) + \cdots) - B(\varepsilon\tau,x,u_0(\varepsilon\tau,x)+\\
&\quad +\varepsilon u_1(\varepsilon\tau,x) + \cdots)\big] + \varepsilon B(t,x,u_0(t,x) + \varepsilon u_1(t,x) + \cdots) \equiv\\
&\equiv \big[A(\varepsilon\tau, x, a_0(\tau,x) + w_0(\tau,x) + \varepsilon(a_1(\tau,x) + w_1(\tau,x)) + \cdots)\frac{\partial}{\partial\tau}(a_0(\tau,x)+\\
&\quad + w_0(\tau,x) + \varepsilon(a_1(\tau,x) + w_1(\tau,x)) + \cdots) - A(\varepsilon\tau, x, a_0(\tau,x)+\\
&\quad +\varepsilon a_1(\tau,x) + \cdots)\frac{\partial}{\partial\tau}(a_0(\tau,x) + \varepsilon a_1(\tau,x) + \cdots)\big] + \varepsilon A(t,x,u_0(t,x)+\\
&\quad +\varepsilon u_1(t,x) + \cdots)\frac{\partial}{\partial t}(u_0(t,x) + \varepsilon u_1(t,x) + \cdots) + \varepsilon\big[B(\varepsilon\tau, x, a_0(\tau,x)+\\
&\quad + w_0(\tau,x) + \varepsilon(a_1(\tau,x) + w_1(\tau,x)) + \cdots) - B(\varepsilon\tau, x, a_0(\tau,x)+\\
&\quad +\varepsilon a_1(\tau,x) + \cdots)\big] + \varepsilon B(t,x,u_0(t,x) + \varepsilon u_1(t,x) + \cdots),
\end{aligned} \tag{8}$$

where

$$a_k(\tau, x) = u_k(0, x) + \tau \frac{\partial u_{k-1}(0, x)}{\partial t} + \cdots + \frac{\tau^k}{k!} \frac{\partial^k u_0(0, x)}{\partial t^k}, \qquad a_k(0, x) = u_k(0, x).$$

From (8) we obtain two types of equations for determining the coefficients $u_k(t, x)$ and $w_k(\tau, x)$, $k \geq 0$, of the expansion (7). For $u_k(t, x)$ we have the following equation:

$$\varepsilon \left(\frac{\partial^2 u_0}{\partial x^2} + \varepsilon \frac{\partial^2 u_1}{\partial x^2} + \cdots \right) - \varepsilon \left(\frac{\partial^2 u_0}{\partial t^2} + \varepsilon \frac{\partial^2 u_1}{\partial t^2} + \cdots \right) =$$
$$= A(t, x, u_0 + \varepsilon u_1 + \cdots) \left(\frac{\partial u_0}{\partial t} + \varepsilon \frac{\partial u_1}{\partial t} + \cdots \right) + B(t, x, u_0 + \varepsilon u_1 + \cdots), \quad (9)$$

and for $w_k(\tau, x)$:

$$\varepsilon^2 \left(\frac{\partial^2 w_0}{\partial x^2} + \varepsilon \frac{\partial^2 w_1}{\partial x^2} + \cdots \right) - \left(\frac{\partial^2 w_0}{\partial \tau^2} + \varepsilon \frac{\partial^2 w_1}{\partial \tau^2} + \cdots \right) = [A(\varepsilon\tau, x, a_0 + w_0 +$$
$$+ \varepsilon(a_1 + w_1) + \cdots) \left(\frac{\partial}{\partial \tau}(a_0 + w_0) + \varepsilon \frac{\partial}{\partial \tau}(a_1 + w_1) + \cdots \right) -$$
$$- A(\varepsilon\tau, x, a_0 + \varepsilon a_1 + \cdots) \left(\frac{\partial a_0}{\partial \tau} + \varepsilon \frac{\partial a_1}{\partial \tau} + \cdots \right)] + \varepsilon [B(\varepsilon\tau, x, a_0 + w_0 +$$
$$+ \varepsilon(a_1 + w_1) + \cdots) - B(\varepsilon\tau, x, a_0 + \varepsilon a_1 + \cdots)]. \quad (10)$$

For the unique determination of the coefficients u_k and w_k of the expansion (7), we prescribe the initial conditions in the following way:

$$u_0(t, x)\big|_{t=0} = \varphi(x) + \Delta(x), \quad (11)$$

$$w_0(\tau, x)\big|_{\tau=0} = \varphi(x) - u_0(0, x), \qquad \frac{\partial w_0}{\partial \tau}\bigg|_{\tau=0} = \psi(x), \quad (12')$$

$$w_k(\tau, x)\big|_{\tau=0} = -u_k(0, x), \qquad \frac{\partial w_k}{\partial \tau}\bigg|_{\tau=0} = -\frac{\partial u_{k-1}(0, x)}{\partial t}, \quad (12'')$$

and the initial conditions for $u_k(t, x)$ $(k > 0)$ will be chosen in a special way (see below).

Expanding the right-hand sides of equations (9) and (10) in powers of ε and then equating coefficients of like powers of ε, we obtain a sequence of equations for $u_k(t, x)$ and $w_k(\tau, x)$:

$$0 = A(t,x,u_0)\frac{\partial u_0}{\partial t} + B(t,x,u_0), \tag{13'}$$

$$\frac{\partial^2 w_0}{\partial \tau^2} + A(0,x,u_0(0,x) + w_0(\tau,x))\frac{\partial w_0}{\partial \tau} = 0, \tag{13''}$$

$$A(t,x,u_0)\frac{\partial u_k}{\partial t} + \left[\frac{\partial A(t,x,u_0)}{\partial u}\frac{\partial u_0}{\partial t} + \frac{\partial B(t,x,u_0)}{\partial u}\right] u_k = \Phi_k(t,x), \tag{14'}$$

$$\frac{\partial^2 w_k}{\partial \tau^2} + \frac{\partial}{\partial \tau}\left[A(0,x,\alpha_0 + w_0)(\alpha_k + w_k) - A(0,x,\alpha_0)\alpha_k\right] = \Psi_k(\tau,x), \tag{14''}$$

where the functions Φ_k and Ψ_k are expressed in terms of $u_i(t,x)$ and $\alpha_i(\tau,x)$, $w_i(\tau,x)$, $i < k$.

The initial conditions for $u_k(t,x)$ $(k > 0)$ have not yet been determined. We shall choose them so that the function $w_k(\tau,x)$ and its first derivative with respect to τ are functions of boundary-layer type (1). In this connection, assuming that w_k, $\partial w_k/\partial\tau$, and $\Psi_k(\tau,x)$ are functions of boundary-layer type, while $A(t,x,u)$, $\partial A/\partial u$ are continuous and $|\partial A/\partial u| < c = \text{const}$ for $t \geq 0$, $-\infty < x < +\infty$, $-\infty < u < +\infty$, we integrate equation (14) with respect to τ from 0 to ∞ and, taking into account

$$\alpha_k(0,x) = u_k(0,x), \qquad u_k(0,x) + w_k(0,x) = 0, \qquad \left.\frac{\partial w_k}{\partial \tau}\right|_{\tau=0} = -\frac{\partial u_{k-1}(0,x)}{\partial t},$$

we obtain the initial condition for $u_k(t,x)$, $k > 0$:

$$u_k(t,x)\big|_{t=0} = \frac{1}{A(0,x,u_0(0,x))}\left(\int_0^\infty \Psi_k(\tau,x)\,d\tau - \frac{\partial u_{k-1}(0,x)}{\partial t}\right). \tag{15}$$

Suppose that in the strip $t \geq 0$, $-\infty < x < +\infty$, for all u the following conditions are satisfied:

a) derivatives of the form

$$\frac{\partial^p A(t,x,u)}{\partial t^{p_0}\partial x^{p_1}\partial u^{p_2}}, \qquad \frac{\partial^p B(t,x,u)}{\partial t^{p_0}\partial x^{p_1}\partial u^{p_2}}, \tag{16}$$

are continuous, where $p = p_0 + p_1 + p_2$, $0 \leq p \leq N+1$, $0 \leq p_i \leq N+1$, $i = 0,1,2$;

b) the following are continuous:

$$\varphi^{(i)}(x), \qquad \psi^{(j)}(x), \tag{17}$$

where $i = 0, 1, \ldots, N+1,\ j = 0, 1, \ldots, N$.

The following assertions are valid.

Lemma 1. The solution $w_0(\tau, x)$ of problem (13), (12) and the derivatives of the form
$\partial^{m+1} w_0 / \partial\tau\, \partial x^m$ are functions of boundary-layer type

$$|w_0(\tau, x)| \le C_0(\tau, x) e^{-\gamma\tau}, \qquad \left| \frac{\partial^{m+1} w_0}{\partial\tau\, \partial x^m} \right| \le C_1(\tau, x) e^{-\gamma\tau}, \tag{18}$$

where $C_0(\tau, x)$ and $C_1(\tau, x)$ are polynomials in τ with bounded coefficients depending on x, $0 \le m \le N+1$.

Proof. Denote $\partial w_0/\partial\tau$ by $v_0(\tau, x)$. Then from (13″), (12′) we have

$$v_0 = \psi(x) \exp\left(- \int_0^\tau A(0, x, u_0(0, x) + w_0)\, ds \right).$$

Hence, bearing (3) in mind, we obtain the estimate (18) for $v_0(\tau, x)$, while

$$w_0(\tau, x) = \varphi(x) - u_0(0, x) + \int_0^\tau v_0(s, x)\, ds,$$

and consequently, as $\tau \to \infty$, we obtain

$$w_0(\infty, x) = \varphi(x) - u_0(0, x) + \int_0^\infty v_0(s, x)\, ds,$$

where $\int_0^\infty v_0(s, x)\, ds$, by virtue of the estimate (18) for $v_0(\tau, x)$, converges. Integrating now equation (13″) and passing to the limit as $\tau \to \infty$, we obtain

$$\psi(x) = \int_{\varphi(x)}^{u_0(0,x) + w_0(\infty, x)} A(0, x, \xi)\, d\xi.$$

Hence, taking into account (6) and the expression for $w_0(\infty, x)$, we obtain

$$\Delta(x) = \int_0^\infty v_0(s, x)\, ds.$$

Then from the expression

$$w_0(\tau, x) = -\int_\tau^\infty v_0(s, x)\, ds$$

directly-

Consequently, estimate (18) follows for $w_0(\tau, x)$. By differentiating equation (13″) and using induction, one can verify the validity of (18) for $\partial^{m+1} w_0(\tau, x)/\partial\tau\,\partial x^m,\ 1 \le m \le N+1$.

Lemma 2. *The solution $w_k(\tau, x)$ of problem (14″), (12″) and its derivatives of the form $\partial^{m+1} w_k/\partial\tau\,\partial x^m,\ 1 \le k \le N+1,\ 0 \le m \le N+1-k$, are functions of boundary-layer type.*

Lemma 2 is proved by the method of mathematical induction.

Theorem. *If conditions (3), (16), (17) are satisfied and, for $-\infty < u < +\infty$, $\partial A/\partial u,\ \partial B/\partial u,\ \partial^2 B/\partial u^2$ are bounded in Q (the characteristic triangle of equation (1)), then in Q there exists a unique*[2,3] *solution of problem (1), (2), and it admits the following asymptotic expansion:*

$$u(t, x, \varepsilon) = \sum_{k=0}^{N} \varepsilon^k u_k(t, x) + \sum_{k=0}^{N+1} \varepsilon^k w_k\left(\frac{t}{\varepsilon}, x\right) + R_N(t, x, \varepsilon), \tag{19}$$

where $u_0(t, x)$ is the solution of equation (4) with initial condition (5), $u_k(t, x)$ is the solution of problem (14′), (15); $w_k(t/\varepsilon, x)$ are functions of boundary-layer type, constructed with the help of problems (13″), (12′) and (14″), (12″). For R_N everywhere in Q the estimate holds

$$\|R_N\|_{L_2(Q)} = O(\varepsilon^{N+1}). \tag{20}$$

Remark. The investigation also carries over to the case $u(t, x_1, \ldots, x_n)$.

In conclusion I express my sincere gratitude to Corresponding Member of the Academy of Sciences of the USSR L. A. Lyusternik for posing the problem and for his constant attention to the work.

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Received
3 I 1970

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Note: Figure translations are in progress. See original paper for figures.

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