THEOREMS ON TRACES AND EXTENSIONS OF FUNCTIONS FROM SURFACES

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Abstract

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M. D. RAMAZANOV

THEOREMS ON TRACES AND EXTENSIONS OF FUNCTIONS FROM SURFACES

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1°. We shall use the following notation: the variables x, y, ξ, η are vectors of the n-dimensional Euclidean space R^n; α, β are multi-indices with integer nonnegative components; iD denotes differentiation with respect to x or y, $i\partial$ with respect to ξ or η. Primed quantities refer to the case of dimension $(n-1)$. For example,

$$D^\alpha = (-i)^{|\alpha|}\partial^{|\alpha|}/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n},$$

$$|\alpha| = \alpha_1 + \dots + \alpha_n, \quad \partial^{\beta'} = (-i)^{|\beta'|}\partial^{|\beta'|}/\partial\xi_1^{\beta_1} \dots \partial\xi_{n-1}^{\beta_{n-1}}.$$

We consider functions of n real variables from the Sobolev spaces $W_2^{\bar{k}}(R^n)$ with norm

$$\|f(x)\|_{\bar{k}} = \left(\int |\tilde{f}(\xi)|^2 \sum_{j=1}^{n} (\xi_j^2 + 1)^{k_j}\, d\xi\right)^{1/2},$$

where

$$\tilde{f}(\xi) = (2\pi)^{-n/2} \int f(x) \exp(-ix\xi)\, dx$$

is the Fourier transform of the function $f(x)$, $\bar{k} = (k_1, \dots, k_n)$.* Under the condition

$$2\min(k_1, \dots, k_n) > \max(1, k_1, \dots, k_n) \tag{1}$$

we describe the properties of the values of functions from $W_2^{\bar{k}}$ on smooth $(n-1)$-dimensional surfaces.

The earlier known results concern mainly the cases when Γ is a hyperplane ([1,2]). Some results for nonplanar surfaces of special types were presented in connection with $W_{p,\alpha}^{\bar{k}}$-spaces in ([3]). Arbitrary smooth surfaces Γ were considered only for isotropic norms ($k_1 = \dots = k_n$) ([1,4]), since in this case Γ can be straightened into a coordinate hyperplane without changing the space.

We use the main result of the work ([5]), which, as applied to the spaces $W_2^{\bar{k}}$ under the condition $2\min(k_1, \dots, k_n) > 1$, is formulated as follows.

Let the $(n-1)$-dimensional surface Γ be given by the functions $x = \varphi(y')$, $y' = (y_1, \dots, y_{n-1}) \in R^{n-1}$; the functions $\varphi_j(y')$ $(j = 1, \dots, n)$ are assumed to be such that there exists a smooth change of variables

$$x = \Phi(y), \tag{2}$$

which maps Γ onto the hyperplane $y_n = 0$, i.e. $\Phi(y', 0) = \varphi(y')$, $\Phi_j(y) \in C^1(R^n)$, and the Jacobian of the change of variables

$$D\Phi = \det(\partial \Phi_i / \partial y_j)$$

is uniformly bounded above and below:

$$0 < c_1 \le D\Phi \le c_2 < \infty.$$

Put

$$P(\xi) = \left[\sum_{j=1}^{n} (\xi_j^2 + 1)^{k_j} \right]^{1/2},$$

$$K(x', y') = \int d\xi \, \{\exp i\xi(\varphi(x') - \varphi(y'))\} / |P(\xi)|^2. \tag{3}$$

* We shall write W_2^k (without a bar over the index k) if $k_1 = \dots = k_n = k$.

Then:

A. The equation

$$\int K(x', y') \, u(y') \, dy' = h(x') \tag{4}$$

is uniquely solvable with respect to $u \in \mathcal{L}_2(R^{n-1})$ for any function $h \in C_0^\infty(R^{n-1})$. With the aid of the inverse operator $u = Lh$, on functions from $C_0^\infty(R^{n-1})$ one can define the norm

$$|||h|||^2 = (Lh, h) > 0 \quad \text{for } h \ne 0. \tag{5}$$

The closure of $C_0^\infty(R^{n-1})$ in this norm gives the space $H(\Gamma)$.

B. Every function $f(x)$ from $W_2^{\bar{k}}(R^n)$ takes on Γ the value $f(\varphi(y')) = h(y') \in H(\Gamma)$. This result is unimprovable in the sense that every function $h(y') \in H(\Gamma)$ can be extended from Γ to all of R^n by a function $v(x) \in W_2^{\bar{k}}(R^n)$.

2°. In the present paper we solve equation (4) under condition (1) and find such an approximate expression for the operator L which permits one to write an explicit expression for the norm equivalent to the norm (5) of the space $H(\Gamma)$ on functions with compact supports. Here the surface Γ is assumed to be sufficiently smooth.

Theorem. *Let*

$$k_0 = \max(k_1, \dots, k_n), \qquad \rho = \min(k_1, \dots, k_n)/k_0,$$

$$\varphi(y') \in C^r, \qquad r = 7k_0 + 2n + n/(2\rho - 1);$$

let $A(y')$ be the value on Γ of the matrix $(\partial\Phi_i/\partial y_j)^{-1}$, let $A^*(y')$ be the transposed matrix, and let

$$p(y', \xi') = \int d\xi_n \,/\, |P(A^*(y')\xi)|^2,$$

where $P(\xi)$ is defined by formula (3). Then, if condition (1) is fulfilled, the norm (5) is equivalent to the norm

$$|||h|||'^2 = \int |h(y')|^2 dy' + \left| \iint dy'\, d\xi' \, \exp(iy'\xi')\, \tilde{h}(\xi')\, h(y')/p(y', \xi') \right|. \quad (6)$$

More precisely, for any compact domain $\Omega \subset R^{n-1}$, the spaces obtained by closing $C_0^\infty(\Omega)$ in the norms (5) and (6) coincide.

Proof. A function concentrated in Ω can be represented as a finite sum of functions with small supports, and it is sufficient to establish the equivalence of the norms (5) and (6) for each summand. Therefore one may regard the domain Ω itself as small. In a sufficiently small domain the equation of the surface Γ can be given as solved with respect to one of the variables. Taking this variable to be x_n, let us define Γ by the equation

$$x_n = \sigma(x').$$

We shall suppose $\sigma(x')$ to be finite, after smoothly deforming, if necessary, Γ into the hyperplane $x_n = 0$ outside some neighborhood of Ω. Obviously, such a deformation will not change the properties of the function in the domain Ω.

We take the change of variables (2) in the form

$$x = \Phi(y) = (y',\, y_n + \sigma(y')).$$

This will allow us to work with less cumbersome formulas, although the proof given is also valid for an arbitrary sufficiently smooth change of variables (2).

In what follows we shall deal only with functions of $n-1$ variables; only the vector ξ will remain n-dimensional. Therefore we agree to omit primes in all notation, except ξ'. Thus, for example, below

$$x = (x_1, \dots, x_{n-1}) \in R^{n-1}, \qquad \alpha = (\alpha_1, \dots, \alpha_{n-1}).$$

Let us write for $\sigma(y)$ the Taylor series in a neighborhood of the point x, with the remainder term in integral form:

$$\sigma(y) = \sigma(x) + \sum_{1 \le |\alpha| \le a} (iD)^\alpha \sigma(x)(y-x)^\alpha/\alpha! + \sigma_a,$$

$$\sigma_a = \int_0^1 dt\,(a+1)(1-t)^a \sum_{|\alpha|=a+1} (iD)^\alpha \sigma(x+t(y-x))(y-x)^\alpha/\alpha!.$$

Corresponding to this, the operator K, defined on $C_0^\infty(\Omega)$ by the formula ··· (3), can be decomposed into the sum of two operators $Kh = K_1 h + K_{ah}$,

$$K_1 h = \int dy\, h(y) \int d\xi \left\{\exp\left[i\left(\xi' + \xi_n \nabla\sigma(x)\right)(x-y)\right] - \right.$$
$$\left. -i\xi_n \sum_{2 \le |\alpha| \le a} (iD)^\alpha \sigma(x)(y-x)^\alpha/\alpha! \right\} / |P(\xi)|^2, \tag{7}$$

where $\nabla\sigma$ is the gradient of the function σ,

$$K_{ah} = \int dy\, h(y) \int d\xi\, (1 - \exp(-i\xi_n \sigma_a))\{\exp[i\xi'(x-y) + i\xi_n(\sigma(y)-\sigma(x))]\}/|P(\xi)|^2.$$

It can be computed that if $a > m - 1 - k_0 + (m+n+1)/\rho$, then

the operator $K_a : W_2^r \to W_2^{r+m/2}$ is bounded for $-m/2 \le r \le m/2$ and $m \ge 2$. (8)

Here we assume $\sigma \in C^{a+1+m}$.

We now consider $K_1 h$, given by formula (7). Denote

$$\Sigma = -\sum_{2 \le |\alpha| \le a} (iD)^\alpha \sigma(x)(y-x)^\alpha/\alpha!.$$

We represent the function $\exp i\xi_n\Sigma$, which stands under the integral in the expression for the kernel of the operator K_1, by Taylor' s formula with the remainder term in integral form,

$$\exp i\xi_n \Sigma = \sum_{j=0}^{b} (i\xi_n \Sigma)^j / j! + e_b.$$

Accordingly, we write K_1 as the sum of two operators: $K_1 h = K_2 h + K_{bh}$, where

$$K_{bh} = \int dy\, h(y) \int \frac{d\xi}{|P(\xi)|^2} \{\exp[i(\xi' + \xi_n \nabla\sigma(x))(x-y)]\} \times$$
$$\times \int_0^1 dt\, \frac{(1-t)^b}{b!}\, (i\xi_n \Sigma)^{b+1} \exp it\xi_n \Sigma.$$

If $b > m + 1 + (\rho - \frac{1}{2})(m + n + 1 - \rho K_0)$ and $\sigma \in C^{a+m}$, then

the operator $K_b : W_2^r \to W_2^{r+m/2}$ is bounded for $-m/2 \le r \le m/2$ and $m \ge 2$. (9)

Let us turn to the operator $K_2 h$:

$$K_2 h = \int dy\, h(y) \int d\xi \left(\sum_{j=0}^{b} \Sigma^j / j! \right) [\exp i(\xi' + \xi_n \nabla\sigma(x))(x-y)]/|P(\xi)|^2 =$$
$$= \sum_{j=0}^{b} [(2\pi)^{n/2}/j!] \int d\xi' \left\{ \int d\xi_n \left(i\xi_n \sum_{2 \le |\alpha| \le a} \frac{(+iD)^\alpha \sigma(x)}{\alpha!} \partial_{\xi'}^\alpha \right)^j \right.$$
$$\left. \times \frac{1}{|P(\xi' - \xi_n \nabla\sigma(x), \xi_n)|^2} \right\} \tilde{h}(\xi') \exp i\xi' x.$$

Bringing under the integral sign the sum over α to the j-th power, we decompose $K_2 h$ into a finite sum of operators corresponding to the individual monomials

$$K_2 h = ph + \sum_{s=1}^{N} c_s(x) p_{sh}, \qquad ph = (2\pi)^{n/2} \int d\xi'\, p(x, \xi') \tilde{h}(\xi') \exp i\xi' x,$$

the coefficients $c_s(x)$ are products of various derivatives of $\sigma(x)$ with constant factors,

$$p_{sh} = \int d\xi'\, p_s(x, \xi') \tilde{h}(\xi') \exp i\xi' x,$$

where the functions $p_s(x, \xi')$ have the form

$$p_s(x, \xi') = \int d\xi_n\, \xi_n^{j_s} \partial_{\xi'}^{\alpha_s} \left(1/|P(\xi' - \xi_n \nabla\sigma(x), \xi_n)|^2\right),$$

$$1 \le j_s \le b, \qquad 2j_s \le |\alpha(s)| \le ab.$$

Lemma 1.

$$\left|\partial^\alpha D^\beta p(x,\xi')\right| \le C_{\alpha,\beta} p(x,\xi')\left(|\xi'|+1\right)^{-\rho|\alpha|+(1-\rho)|\beta|},$$

$$\left|\partial^\alpha D^\beta p_s(x,\xi')\right| \le C_{\alpha,\beta,s} p(x,\xi')\left(|\xi'|+1\right)^{-\rho|\alpha|+(1-\rho)|\beta|-(\rho-1)s}.$$

These estimates are not difficult to obtain, taking into account the simple expressions of the functions $p(x,\xi')$, $p_s(x,\xi')$ in terms of $P(\xi)$ and the definition of $P(\xi)$.

Let

$$qh = \zeta(x)\int d\xi' \int dy\,|\zeta(y)h(y)\exp i\xi'(x-y)|/p'(x,\xi'),$$

where $\zeta(x) \in C_0^\infty(R^{n-1})$ and is equal to 1 on Ω.

Lemma 2. If $\sigma \in C^r,\ r = 7k_0 + 2n + n/(2\rho-1)$, then the operator $q : W_2^r \to W_2^{r-k_0}$ is bounded for $0 \le r \le k_0$, $qK = \xi^2 + q_1,\ Kq = \xi^2 + q_2$, where the operator $\xi^2 h = \xi^2(x)h(x)$, and $q_1 : W_2^r \to W_2^{r+2\rho-1}$ is bounded for $-k_0 \le r \le 0$, the operator $q_2 : W_2^r \to W_2^{r+2\rho-1}$ is bounded for $0 \le r \le k_0$.

This assertion can be established by the method developed in Hörmander's paper (6) (proof of Theorem 3.1). In doing so we use the decomposition obtained by us

$$K = p + \sum c_{sp}s + K_a + K_b.$$

For the operators $q(K_a + K_b)$ and $(K_a + K_b)q$, the boundedness of their action from W_2^r to $W_2^{r+2\rho-1}$ for $-k_0 \le r \le 0$ and $0 \le r \le k_0$, respectively, follows from formulas (8) and (9) with $m/2 = k_0 + 2\rho - 1,\ a = 5k_0 + 2n - 1,\ b = 3k_0 + 4 + (n+1)/2$. The action of the operators q, p, p_s is studied on the basis of the estimates of Lemma 1.

Lemma 2 means that the operator q may be taken approximately as the inverse of K, the operator L.

We complete the proof of the theorem. On functions concentrated in Ω,

$$L = L\zeta^2 = L(Kq - q_2) = q - Lq_2, \qquad \|h\|^2 = (Lh,h) = (qh,h) - (Lq_2h,h).$$

If $\sup h \subset \omega$, then

$$|(qh,h)| = (2\pi)^{-n/2} \left| \int dx\, d\xi' \, \exp(i\xi' x)\, \tilde{h}(\xi')\overline{h}(x)/p(x,\xi') \right|.$$

The equivalence of the norms (5) and (6) will be established if we show that

$$|(Lq_2 h, h)| \le \|h\|^2/2 + C\|h\|^2_{\mathcal{L}_2}. \tag{10}$$

The last estimate is indeed valid. Note that (Lh, g) defines the scalar product in the Hilbert space $H(\Gamma)$. For any $\delta > 0$ we have

$$|(Lq_2 h, h)| \le \|h\| \cdot \|q_2 h\| \le \delta\|h\|^2 + (4\delta)^{-1}\|q_2 h\|^2 =$$

$$= \delta\|h\|^2 + (4\delta)^{-1}(LKq_2^* Lq_2 h, h) \le 2\delta\|h\|^2 + (4\delta)^{-3}\|Kq_2^* Lq_2 h\|^2 \le \cdots$$

$$\cdots \le s\delta\|h\|^2 + (4\delta)^{1-2s}\|(Kq_2^* Lq_2)^{2s-1} h\|^2,$$

$$Lq_2 = L - q = q_1 L, \qquad Kq_2^* L = q_1^*.$$

Thus,

$$|(Lq_2 h, h)| \le s\delta\|h\|^2 + (4\delta)^{1-2s}\|(q_1^* q_2)^{2s-1} h\|^2. \tag{11}$$

The operator $(q_1^* q_2)^{2s-1} : \mathcal{L}_2 \to W_2^{k_0}$ is bounded for $s \ge \ln(k_0/(2\rho - 1))/\ln 2$. Consequently, we obtain inequality (10) by putting in (11)

$$s = [\ln(k_0/(2\rho - 1))/\ln 2] + 1$$

and $\delta = 1/2s$.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR
Novosibirsk

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Note: Figure translations are in progress. See original paper for figures.