DECOMPOSITION OF THE TENSOR PRODUCT OF IRREDUCIBLE MAXIMALLY DEGENERATE REPRESENTATIONS OF THE GROUP OF COMPLEX MATRICES OF ORDER \(n\) WITH DETERMINANT 1

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DECOMPOSITION OF THE TENSOR PRODUCT OF IRREDUCIBLE MAXIMALLY DEGENERATE REPRESENTATIONS OF THE GROUP OF COMPLEX MATRICES OF ORDER n WITH DETERMINANT 1

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In representation theory, in connection with various applications, in particular in theoretical physics, the problem arises of decomposing the tensor product of irreducible representations of a certain group into irreducible representations of this group. For the group of complex matrices of order n with determinant 1, the problem was solved by I. M. Gel' fand and M. I. Graev $(^4)$ in the case of the tensor product of representations of the principal series. For $n = 2$ an analogous problem was solved by another method by M. A. Naimark $(^3)$. For $n = 2$ the problem was solved by M. A. Naimark for the case of the tensor product of representations of the principal and supplementary series and for the case of the tensor product of 2 representations of the supplementary series $(^{5,6})$. In the present article the problem is solved of decomposing into irreducible representations the tensor product of irreducible maximally degenerate representations of the group of complex matrices of order n with determinant 1. In doing so, a modified method of I. M. Gel' fand and M. I. Graev $(^4)$ of transformation by horospheres is used.

1. Statement of the problem. Let G be the group of complex matrices of order n with determinant 1 $(n \geq 3)$. Let K be the group of matrices $k = \|k_{pq}\| \in G$ such that $k_{p1} = 0,\ p \geq 2$. Let Z be the group of matrices $z = \|z_{pq}\| \in G$ such that $z_{pp} = 1,\ z_{pq} = 0,\ q \neq 1,\ q \neq p$. It is known $(^1)$ that almost any matrix $g \in G$ can be represented in the form $g = k \cdot z$, where $k \in K,\ z \in Z$. Recall $(^1)$ that the maximally degenerate representation $\mathfrak{S}_{p,\rho}$ (p an integer, ρ a real number) of the group G is realized in the space of functions on the group Z, $f(z) = f(z_{21}, \dots, z_{n1})$, with integrable square and is given by the formula

$$T_g f(z) = \alpha(zg) f(\widetilde{zg}), \tag{1}$$

where

$$zg = k \cdot \widetilde{zg}, \qquad k \in K, \qquad \widetilde{zg} \in Z,$$

$$\alpha(g) = \alpha(k \cdot z) = \alpha(k) = |k_{11}|^{-p-i\rho+n} k_{11}^{p}. \tag{2}$$

The tensor product $\mathfrak{S}_{p_1,\rho_1} \times \mathfrak{S}_{p_2,\rho_2}$ of two representations $\mathfrak{S}_{p_1,\rho_1}, \mathfrak{S}_{p_2,\rho_2}$ is realized in the space $L^2(Z \times Z)$ of functions $f(z, z') = f(z_{21}, \dots, z_{n1}, z'_{21}, \dots, z'_{n1})$ on the direct product $Z \times Z$, having integrable square, and is given by the formula

$$T_g f(z, z') = \alpha_1(zg)\alpha_2(z'g) f(\widetilde{zg}, \widetilde{z'g}), \tag{3}$$

where

$$\alpha_1(k) = |k_{11}|^{-p_1-i\rho_1+n} k_{11}^{p_1}, \qquad \alpha_2(k) = |k_{11}|^{-p_2-i\rho_2+n} k_{11}^{p_2+i\rho_2}.$$

It is required to decompose the representation (3) into irreducible representations.

2. The manifold of pairs (z, z') is not transitive with respect to the group G. However, it is easy to show that after removing from it a submanifold of lower dimension, a transitive manifold $Y = G/C$ is obtained, where

C is the group of matrices $c = \|c_{pq}\| \in G$ such that $c_{p1} = 0,\ p \geq 2;\ c_{12} = 0,\ c_{p2} = 0,\ p \geq 3$. Let U be the group of matrices $u = \|u_{pq}\| \in G$ such that $u_{pp} = 1,\ u_{pq} = 0,\ p < q;\ u_{pq} = 0,\ 2 < q < p;\ \Xi$ the group of matrices $\xi = \|\xi_{pq}\| \in G$ such that $\xi_{pp} = 1,\ \xi_{pq} = 0,\ (p, q) \neq (1, 2)$, and M the group of matrices $m = \|m_{pq}\| \in G$ such that $m_{p1} = 0,\ p \geq 2;\ m_{p2} = 0,\ p \geq 3$.

Then (^{1}) almost every matrix $g \in G$ can be represented in the form

$$g = mu = c\xi u, \qquad m \in M, \qquad u \in U, \qquad \xi \in \Xi, \qquad c \in C. \tag{4}$$

By virtue of (4), to each pair (ξ, u), $\xi \in \Xi$, $u \in U$, there corresponds a right adjacency class in Y. The points in Y for which there is no such correspondence form in Y a submanifold of lower dimension. Define on Y the measure

$$dy = d\xi\, du = d\xi_{12} du_{21} \cdots du_{n1} du_{32} \cdots du_{n2}. \tag{5}$$

Consider the representation of the group G which is realized in the space $L^2(Y)$ of functions $f(y)$ on the manifold Y, square-integrable with respect to the measure dy, and which is given by the formula

$$T_{gf}(y) = T_{gf}(\xi, u) = \alpha(ug) f(yg), \tag{6}$$

where

$$\alpha(g) = \alpha(mu) = \alpha(m) = |m_{11}|^{-p_1 - i\rho_1 + n} m_{11}^{p_1} |m_{22}|^{-p_2 - i\rho_2 + n} m_{22}^{p_2}.$$

Let $f(z, z')$ be a finite function equal to 0 in a neighborhood of the hyperplane $z_{21} = z'_{21}$. Put it in correspondence with the function $\varphi(y) \in L^2(Y)$ by the formula:

$$\varphi(\xi_{12}, u_{21}, \dots, u_{n2}) = |\xi_{12}|^{p_2 + i\rho_2 - n} \xi_{12}^{-p_2} \times$$

$$\times f\left(u_{21},\ u_{31} - u_{32}u_{21}, \dots,\ u_{n1} - u_{n2}u_{21},\ u_{21} + \frac{1}{\xi_{12}},\right.$$

$$\left. u_{31} - u_{32}u_{21} - \frac{u_{32}}{\xi_{12}}, \dots,\ u_{n1} - u_{n2}u_{21} - \frac{u_{n2}}{\xi_{12}}\right). \tag{7}$$

Then the inverse formula has the form

$$f(z, z') = |z'_{21} - z_{21}|^{p_2 + i\rho_2 - n} (z'_{21} - z_{21})^{-p_2} \times$$

$$\times \varphi\left(\frac{1}{z'_{21} - z_{21}},\ z_{21},\ \frac{z_{31}z'_{21} - z'_{31}z_{21}}{z'_{21} - z_{21}}, \dots,\ \frac{z_{n1}z'_{21} - z'_{n1}z_{21}}{z'_{21} - z_{21}},\right.$$

$$\left. \frac{z_{31} - z'_{31}}{z'_{21} - z_{21}}, \dots,\ \frac{z_{n1} - z'_{n1}}{z'_{21} - z_{21}}\right). \tag{8}$$

Theorem 1. The mapping (7) defines an isomorphism of the spaces $L^2(Z \times Z')$ and $L^2(Y)$. In this case the representations given by formulas (3) and (6) are equivalent.

3. Consider the group H of matrices $h = \|h_{pq}\| \in G$ such that $h_{22} = 1,\ h_{pp} = 1,\ p \geq 4;\ h_{pq} = 0,\ q > p;\ h_{pq} = 0,\ 3 \leq q < p$. Let Σ be the group of diagonal matrices such that $\sigma_{22} = 1,\ \sigma_{pp} = 1,\ p \geq 4$, and Λ the group of matrices $\lambda = \|\lambda_{pq}\| \in M$ such that $\lambda_{11} = \lambda_{22}$. Then (1) almost every $g \in G$ can be represented in the form

$$g = \lambda h = \lambda \sigma u, \qquad \lambda \in \Lambda, \qquad h \in H, \qquad \sigma \in \Sigma, \qquad u \in U.$$

The group G acts on the group H. Define on H a right-invariant measure

$$d_{rh} = |\sigma|^{-2(n+2)} d\sigma\, du = |\sigma|^{-2(n+2)} d\sigma\, du_{21} \cdots du_{n2}.$$

Consider the representation of the group G which is realized in the space $L^2(H)$ of functions $f(h)$ on H, square-integrable with respect to the measure d_{rh}, and is given by the formula

$$T_{gf}(h) = \gamma(hg) f(\overline{hg}), \tag{9}$$

where

$$hg = \overline{\lambda hg}, \qquad \lambda \in \Lambda, \qquad \overline{hg} \in H, \tag{10}$$

$$\gamma(g) = \gamma(\lambda h) = \gamma(\lambda) = |\lambda_{22}|^{-p-i\rho+2n} \lambda_{22}^{p} \exp[-i \operatorname{Re}(\lambda_{12}/\lambda_{22})].$$

Consider the mapping of the space $L^2(Y)$ into the space $L^2(H)$

$$f(\sigma, u) = |\sigma|^{-p-i\rho_1+n+2\sigma p_1} \int \exp[i \operatorname{Re} \sigma \xi_{12}] \varphi(\xi, u)\, d\xi_{12}. \tag{11}$$

The inversion formula has the form

$$\varphi(\xi, u) = \frac{1}{2\pi} \int |\sigma|^{p_1+i\rho_1-n-2} \sigma^{-p_1} f(\sigma, u) \exp[-i \operatorname{Re} \sigma \xi_{12}]\, d\sigma. \tag{12}$$

Theorem 2. The mapping (11) defines an isomorphism of the spaces $L^2(Y)$ and $L^2(H)$. The representations defined by formulas (6) and (9) are equivalent when $p_1 + p_2 = p,\ \rho_1 + \rho_2 = \rho$.

Corollary. If $p_1 + p_2 = p_1' + p_2'$, $\rho_1 + \rho_2 = \rho_1' + \rho_2'$, then the tensor products $\mathfrak{S}_{p_1,\rho_1} \times \mathfrak{S}_{p_2,\rho_2}$ and $\mathfrak{S}_{p_1',\rho_1'} \times \mathfrak{S}_{p_2',\rho_2'}$ are unitarily equivalent.

4. In order to decompose the representation obtained in § 3 in $L^2(H)$, it is necessary in formula (10) to get rid of the exponent. For this we use the transformation with respect to the orispherical subgroup Ξ. Let $z_0 = \|z_{pq}\| \in G$ be a matrix such that $z_{pp} = 1,\ p \geq 3;\ z_{11} = z_{22} = 0;\ z_{21} = -z_{12} = 1$; the remaining $z_{pq} = 0$. Consider the mapping of $L^2(H)$ onto itself

$$\check{f}(h) = \frac{1}{\pi} L_1 \int \gamma(z_0 \xi h) f(\overline{z_0 \xi h})\, d\xi_{12}, \tag{13}$$

where

$$L_1 = 2\sigma \frac{d}{d\sigma} + \left(\frac{-p + i\rho}{2} - n - 1 \right), \qquad \overline{z_0 \xi h} \in H, \quad \xi \in \Xi.$$

The inversion formula has the form

$$f(\sigma, u) = -\frac{1}{4} |\sigma|^{(-p-i\rho)/2+n+1} \sigma^{p/2} \int \exp[i \operatorname{Re} \sqrt{\sigma}\, |t|]\, |t|^{(p+i\rho)/2-n-3} t^{-p/2} \times$$

$$\times [L_2 \check{f}](t, \sqrt{\sigma t} + u_{21}, u_{31}, \dots, u_{n2}), \tag{14}$$

where $L_2 = 2\bar{\sigma}\, \partial/\partial\bar{\sigma} + ((p + i\rho)/2 - n - 1)$. The Plancherel formula has the form

$$\frac{1}{4\pi^2} \int |f(\sigma, u)|^2 |\sigma|^{-2n-4}\, d\sigma\, du = \frac{1}{8\pi^2} \int |\check{f}(\sigma, u)|^2 |\sigma|^{-2n-4}\, d\sigma\, du.$$

The representation which was defined by formula (9), after the transformation will be defined by the following formula:

$$T_g \check{f}(h) = \varepsilon(hg) \check{f}(\overline{hg}), \tag{15}$$

where $\varepsilon(g) = \varepsilon(\lambda h) = \varepsilon(\lambda) = |\lambda_{22}|^{-p-i\rho+2n} \lambda_{22}^{p}$.

Theorem 3. The mapping (13) is an isometric mapping of the space $L^2(H)$ onto itself. Moreover, the representation defined by formula (9) is equivalent to the representation defined by formula (15).

5. Let $\check{f}(h) \in L^2(H)$. Consider the transformation

$$\varphi(u, \chi) = \int \check{f}(\sigma, u) \chi^{-1}(\sigma) |\sigma|^{-n-1}\, d\mu(\sigma),$$

where $\chi(\sigma) = |\sigma|^{l+iv} \sigma^{-l}$ is a character on the group Σ, l is an integer, v is a real number, and $d\mu(\sigma)$ is an invariant measure on the group Σ. The inversion and Plancherel formulas have the form

$$\check{f}(\sigma, u) |\sigma|^{-n-1} = \frac{1}{2\pi^2} \sum_{l=-\infty}^{\infty} \int_0^{\infty} \varphi(u, l, v) \chi(\sigma)\, dv,$$

$$\frac{1}{8\pi^2}\int |\check{f}(\sigma,u)|^2|\sigma|^{-2n-2}\,d\mu(\sigma)\,du = \frac{1}{(2\pi)^4}\sum_{l=-\infty}^{\infty}\int_0^{\infty}\left[\int |\varphi(u,l,v)|^2\,du\right]dv.$$

The operators of the representation take the form

$$T_g\varphi(u,\chi) = a_\chi(ug)\varphi(\widehat{ug},\chi), \qquad ug = m\widehat{ug}, \qquad m\in M, \qquad \widehat{ug}\in U,$$

where

$$a_\chi(g) = a_\chi(mu) = a_\chi(m) = |m_{11}|^{l+i\nu+n+1}m_{11}^{-l}|m_{22}|^{-(p+l)-i(\rho+\nu)+n-1}m_{22}^{p+l}.$$

Thus, for fixed χ, the functions $\varphi(u,\chi)$ are transformed according to the unitary degenerate representation of the group G corresponding to the character χ, with stationary subgroup M. Using this and Theorems 1-3, we obtain the main theorem.

Theorem 4. Let L denote the Hilbert space of all measurable functions $\varphi(u,\chi)$ satisfying the condition

$$\|\varphi(u,\chi)\|^2 = \sum_{l=-\infty}^{\infty}\int_0^{\infty}\omega(\chi)\left(\int|\varphi(u,\chi)|^2\,du\right)d\nu < \infty,$$

where

$$\omega(\chi) = \frac{1}{(2\pi)^4}\left|\rho_1+\rho_2+2l-i(\rho_1+\rho_2+2\nu)\right|^2.$$

Using δ-functions, we have that for any function $f(z,z')\in L^2(Z\times Z)$ the integral

$$\varphi(u,\chi) = \int b(u,z,z',\chi)f(z,z')\,dz\,dz', \tag{16}$$

where

$$\begin{aligned} b(u,z,z',\chi) = {} & |z'_{21}-u_{21}|^{-(l+p_2)-i(\nu+\rho_2)-1}(z'_{21}-u_{21})^{l+p_2}\times \\ & \times|z_{21}-u_{21}|^{-(l+p_1)-(\nu+\rho_1)-1}(z_{21}-u_{21})^{l+p_1}|z'_{21}-z_{21}|^{l+i\nu+n-3}\times \\ & \times(z'_{21}-z_{21})^{-l}\delta[u_{31}-u_{32}z_{21}-z_{31}]\times\cdots\times\delta[u_{n1}-u_{n2}z_{21}-z_{n1}] \\ & \times\delta[u_{31}-u_{32}z'_{21}-z'_{31}]\times\cdots\times\delta[u_{n1}-u_{n2}z'_{21}-z'_{n1}] \end{aligned}$$

converges in the sense of the norm in L, and the correspondence determined by this integral is an isometric mapping of the space $L^2(Z \times Z)$ onto the space L. The inverse mapping is given by the formula

$$f(z, z') = \int \overline{b(u, z, z', \chi)}\, \varphi(u, \chi)\omega(\chi)\, d\chi, \qquad d\chi = d\nu.$$

When passing from $f(z, z')$ to $T_g f(z, z')$, the corresponding function $\varphi(u, \chi)$ passes to $T_g\varphi(u, \chi)$; consequently, the mapping (16) realizes the decomposition of the representation $\mathfrak{S}_{p_1,\rho_1} \times \mathfrak{S}_{p_2,\rho_2}$ into irreducible representations.

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Note: Figure translations are in progress. See original paper for figures.