ON CONJUGATE HARMONIC FUNCTIONS OF SEVERAL VARIABLES

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Abstract

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ON CONJUGATE HARMONIC FUNCTIONS OF SEVERAL VARIABLES

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Denote by $\mathbf{x} = (x_1, \dots, x_n)$ the points of n-dimensional Euclidean space $\mathbb{R}^n$; y is a real variable; $(\mathbf{x}, y) = (x_1, \dots, x_n, y)$ are the points of the Cartesian product $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$; $\mathbf{t} \cdot \mathbf{x} = t_1 x_1 + \cdots + t_{nx} n$; $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$; dt is the element of Lebesgue measure in $\mathbb{R}^n$. In the half-space $\mathbb{R}^{n+1}+ = \mathbb{R}^n \times (0, +\infty)$ we shall consider harmonic vectors $\mathbf{F}(\mathbf{x}, y) = (U, V_1, \dots, V_n)$, i.e. ([1]) vector-functions $(\mathbf{x}, y)$, whose components are harmonic functions satisfying the generalized Cauchy–Riemann conditions

$$\frac{\partial U}{\partial y} + \sum_{k=1}^{n} \frac{\partial V_k}{\partial x_k} = 0, \quad \frac{\partial U}{\partial x_k} = \frac{\partial V_k}{\partial y}, \quad \frac{\partial V_k}{\partial x_j} = \frac{\partial V_j}{\partial x_k}, \quad k \neq j;\ k, j = 1, 2, \dots, n.$$

In the present paper the theorems proved by Hardy, Littlewood ([2]) and Kawata ([3]) in the plane are generalized to the multidimensional case.

Denote

$$M(y) = M(\mathbf{F}, y) = \max_{\mathbf{x} \in \mathbb{R}^n} |\mathbf{F}(\mathbf{x}, y)|; \tag{1}$$

$$M_p(y) = M_p(\mathbf{F}, y) = \left\{ \int_{\mathbb{R}^n} |\mathbf{F}(\mathbf{x}, y)|^p \, dx \right\}^{1/p}. \tag{2}$$

Definition 1. We shall say that a harmonic vector $\mathbf{F}(\mathbf{x}, y)$ in $\mathbb{R}^{n+1}+$ **belongs to the class** H^p, $p > 0$, if $M_p(\mathbf{F}, y) < C$, where C does not depend on y.

Definition 2. We shall say that a harmonic vector $\mathbf{F}(\mathbf{x}, y)$ in $\mathbb{R}^{n+1}+$ **belongs to the class** S^p, $p > 0$, if for every $y_0 > 0$ there exists a constant $C(y_0, \mathbf{F})$ such that, for all $y \geq y_0$, $M_p(\mathbf{F}, y) \leq C(y_0, \mathbf{F})$.

As is known ([1]), for $p \geq (n-1)/n$ harmonic vectors $\mathbf{F}(\mathbf{x}, y) \in H^p$ have finite boundary values $f(\mathbf{x})$ almost everywhere in $\mathbb{R}^n$ as $y \downarrow 0$, and for $p > (n-1)/n$ convergence in the mean with exponent p also holds.

Theorem 1. Let $f(\mathbf{x}) = (u, v_1, \dots, v_n) \in L^2(\mathbb{R}^n)$. Then the necessary and sufficient condition for $f(\mathbf{x})$ to be the boundary value as $y \downarrow 0$ of a harmonic vector $\mathbf{F}(\mathbf{x}, y)$ of class H^2 in $\mathbb{R}^{n+1}+$ is that the Fourier transform of the function $f(\mathbf{x})$ can be represented in the form

$$g(\mathbf{x}) \left(e_0 + i \sum_{k=1}^{n} e_k \frac{t_k}{|\mathbf{t}|} \right), \tag{3}$$

where $e_0, e_1, \dots, e_n$ are the units of the Clifford algebra, and

$$g(\mathbf{x}) = \underset{T \to \infty}{\text{l. i. m.}} \frac{1}{(2\pi)^{n/2}} \int_{|\mathbf{t}|<T} u(\mathbf{t}) e^{i\mathbf{x}\cdot\mathbf{t}}\, dt.$$

The proof is obtained by applying the Plancherel theorem to the representation of $\mathbf{F}(\mathbf{x}, y)$ by means of the Poisson integral. As a result one also obtains the following integral representation for harmonic vectors of class $\mathbf{H}^2$ in $\mathbf{R}_+^{n+1}$

$$\mathbf{F}(\mathbf{x}, y) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} g(\mathbf{t}) e^{-y|\mathbf{t}|} \left(e_0 + i \sum_{k=1}^{n} e_k \frac{t_k}{|\mathbf{t}|} \right) e^{-i\mathbf{x}\cdot\mathbf{t}}\, d\mathbf{t}. \tag{4}$$

Lemma. If $p \geq (n-1)/n,\ a \geq 0,\ \mathbf{F}(\mathbf{x}, y) = (U, V_1, \dots, V_n)$ is a harmonic vector of class S^p in $\mathbf{R}_+^{n+1}$, then from the inequality

$$M_p(U, y) \leq C y^{-a} \tag{5}$$

it follows that

$$|\mathbf{F}(\mathbf{x}, y)| \leq BCy^{-B}, \tag{6}$$

where B and C are positive constants.

Proof. We first consider the case $(n-1)/n \leq p \leq 1$. Choose $C = 1,\ y_0 > 0$, and consider the harmonic vector $(\mathbf{x}, y) = \mathbf{F}(\mathbf{x}, y + y_0)$. According to (4), $(\mathbf{x}, y) = (W_0, W_1, \dots, W_n)$ belongs to $\mathbf{H}^1 \cap \mathbf{H}^2$ in $\mathbf{R}_+^{n+1}$. Let $G(\mathbf{t})$ be the Fourier transform of the boundary values $W_0(\mathbf{x})$ of the function $W_0(\mathbf{x}, y)$. Then

$$G(\mathbf{t}) e^{-y(\mathbf{t})} = \frac{1}{(2\pi)^{n/2}} \int_{R^n} W_0(\mathbf{x}, y) e^{i\mathbf{x}\cdot\mathbf{t}}\, d\mathbf{t}. \tag{7}$$

If $p = 1$, then from (4) and (5) it follows that

$$|\mathbf{F}(\mathbf{x}, y+y_0)| \leq \frac{2}{(2\pi)^{n/2}} \int_{R^n} |G(\mathbf{t})| e^{-(y+y_0)|\mathbf{t}|}\, d\mathbf{t} \leq \frac{2}{(2\pi)^n} (y+y_0)^{-a} \int_{R^n} e^{-y_0|\mathbf{t}|}\, d\mathbf{t},$$

or $|\mathbf{F}(\mathbf{x}, y+y_0)| = O((y+y_0)^{-a-n})$.

If $(n-1)/n \leq p < 1$, then from (7) and (5) we have

$$|G(\mathbf{t})| e^{-y|\mathbf{t}|} \leq \frac{1}{(2\pi)^{n/2}} M^{1-p}(U, \eta) \eta^{-ap}, \qquad \eta = y + y_0. \tag{8}$$

Let $\eta_1 = y_1 + y_0 > y + y_0 = \eta$. Then

$$|\mathbf{F}(\mathbf{x}, \eta)| \leq \frac{2}{(2\pi)^{n/2}} M^{1-p}(U, \eta) \int_{R^n} e^{-(y_1-y)|\mathbf{t}|}\, d\mathbf{t}, \tag{9}$$

$$|\mathbf{F}(\mathbf{x}, \eta)| \leq K M^{1-p}(\mathbf{F}, \eta) \eta^{-ap} (\eta_1 - \eta)^{-n}, \qquad K = \text{const}.$$

Using the results of the paper ([2]), from (9) we obtain

$$M(\mathbf{F}, y) \leq B y^{-B}.$$

Theorem 2. If a harmonic vector $\mathbf{F}(\mathbf{x}, y) = (U, V_1, \dots, V_n)$ of class S^p in $\mathbf{R}_+^{n+1}$, $p \geq (n-1)/n$, $a \geq 0$, $q > p$, then from

$$M_p(U, y) \leq C y^{-a} \tag{10}$$

it follows that

$$M_q(\mathbf{F}, y) \leq BC y^{-a-n/p+n/q}. \tag{11}$$

In particular, for $q = \infty$,

$$M(\mathbf{F}, y) \leq BC y^{-a-n/p}. \tag{12}$$

Here B, C are constants independent of y.

Proof. We shall first carry it out for $(n-1)/n \leq p \leq 1$. It suffices to prove (12). Put $\tau = 1/\eta_1$, $\quad t = 2/\eta_1 = 1/\eta$, $\quad h(\tau) = \ln(\eta_1^{a+n/p} M(\eta_1))$, $\quad b = 1 - p$.

Then, using (9),

$$h(\tau) - bh(2\tau) \leq (a + n/p) \ln(\eta_1/\eta) + n \ln(\eta/(\eta_1 - \eta)) < B.$$

On the basis of (2), either $h(\tau)$ is bounded by a constant, and the theorem is true, or

$$\overline{\lim_{\tau\to\infty}}\, h(\tau)/\ln\tau = \infty,$$

but the latter contradicts the lemma.

The case $p > 1$ is obtained with the aid of the following two theorems.

Theorem 3. Let $\mathbf{F}(\mathbf{x}, y) = (U, V_1, \dots, V_n)$ be a harmonic vector of class $S^p(\mathbf{R}^{n+1}+)$, $p > 1$. There exists a constant C_p, depending only on p, such that for every $y > 0$ the inequality

$$\|\mathbf{F}(\mathbf{x}, y)\|_p \leq C_p \|U(\mathbf{x}, y)\|_p$$

holds.

Theorem 4. If $p \geq (n-1)/n$, $a \geq 0$, $q > p$, and a harmonic vector $\mathbf{F}(\mathbf{x}, y)$ belongs to the class S^p in $\mathbf{R}^{n+1}+$, then from the condition

$$M_p(\mathbf{F}, y) = O(y^{-a}) \tag{13}$$

it follows that

$$M_q(\mathbf{F}, y) = O(y^{-a-n/p+n/q}). \tag{14}$$

In particular, for $q = \infty$,

$$M(\mathbf{F}, y) = O(y^{-a-n/p}). \tag{15}$$

Proof for $(n-1)/n \leq p \leq 1$ follows from the part of the proof of Theorem 2 already carried out. Let $p > 1$ and take $\eta > 0$. In the half-space $\mathbf{R}^n \times (\eta, +\infty)$ the function $\mathbf{F}(\mathbf{x}, y) \in H^p$, and therefore, by (1), is representable by the Poisson-Lebesgue integral

$$\mathbf{F}(\mathbf{x}, y) = \frac{1}{c_n} \int_{R^n} \frac{\mathbf{F}(\mathbf{t}, \eta)(y-\eta)}{\left(|\mathbf{x}-\mathbf{t}|^2 + (y-\eta)^2\right)^{(n+1)/2}}\, d\mathbf{t}, \qquad c_n = \frac{\pi^{(n+1)/2}}{\Gamma((n+1)/2)}.$$

With the aid of Hölder' s inequality we have

$$|\mathbf{F}(\mathbf{x}, y)| \leq M_p(\mathbf{F}, \eta) \frac{1}{c_n} \left\{ \int_{R^n} \frac{(y-\eta)^{p'}}{\left(|\mathbf{x}-\mathbf{t}|^2 + (y-\eta)^2\right)^{(n+1)p'/2}}\, d\mathbf{t} \right\}^{1/p'},$$

where $p' = p/(p-1)$, and hence

$$|\mathbf{F}(\mathbf{x}, y)| = O(y^{-a-n/p}).$$

To prove (11), we note on the basis of (10) that

$$M_q(\mathbf{F}, y) \le M^{(q-p)/q}(\mathbf{F}, y)\, M_p^{p/q} \le B y^{-a-n/p+n/q}.$$

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Note: Figure translations are in progress. See original paper for figures.