ON THE CHANGE OF THE ADIABATIC INVARIANT OF A NONLINEAR PERIODIC WAVE

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ON THE CHANGE OF THE ADIABATIC INVARIANT OF A NONLINEAR PERIODIC WAVE

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The behavior of dynamical systems with parameters slowly varying in time has been studied rather well. There exists, on the one hand, a proof of the conservation of adiabatic invariants in all orders of perturbation theory ([1]), and, on the other hand, sufficiently convenient methods for calculating exponentially small corrections to adiabatic invariants ([2,3]). One should also mention the more universal theorems of Arnold–Kolmogorov ([4]) and Moser ([5]).

The subject of this article is the study of nonlinear periodic waves described by partial differential equations in a weakly inhomogeneous medium. At present there is only one result of Moser ([6]), the content of which is a proof of conservation, in all orders of perturbation theory, of a certain quantity for a nonlinear stationary wave, analogous to the adiabatic invariant of a dynamical system. We shall show below how exponentially small corrections to the quantities characterizing the wave can be calculated.

For definiteness, let us consider a model in which nonlinear stationary waves are described by the Korteweg–de Vries equation:

$$v_t + vv_x + \Lambda v_{xxx} = 0, \tag{1}$$

where $\Lambda = \Lambda(x)$ is a function slowly depending on the coordinate and characterizing the inhomogeneity of the medium. Equation (1) is obtained in various problems of plasma theory and waves on water of finite depth ([7]), oscillations of a chain of coupled oscillators ([8]), etc.

Equation (1) can be replaced by the canonical equations of motion

$$\dot{v}(q) = iq\,\frac{\delta H(\Lambda)}{\delta v(-q)}; \qquad v = \int dq\, e^{iqx} v(q); \qquad v(-q) = v^*(q), \tag{2}$$

where the Hamiltonian $H(\Lambda)$ is defined by the expression

$$H(\Lambda) = -\frac{1}{2}\int dq_1 dq_2 dq_3\, q_1 q_2 v(q_1) v(q_2) \Lambda(q_3) \delta(q_1+q_2+q_3) -$$

$$-\frac{1}{6}\int dq_1 dq_2 dq_3\, v(q_1) v(q_2) v(q_3) \delta(q_1+q_2+q_3). \tag{3}$$

In the case when $\Lambda(x) = \Lambda_0$ and Λ_0 does not depend on x, the Hamiltonian $H_0 \equiv H(\Lambda_0)$ for a nonlinear periodic wave degenerates into the sum

$$H_0 = \frac{1}{2}\Lambda_0 \sum_n (kn)^2 v_{nv_{-n}} - \frac{1}{6}\sum_{n_1+n_2+n_3=0} v_{n_1} v_{n_2} v_{n_3}, \tag{4}$$

where $\lambda = 2\pi/k$ is the spatial period of the wave oscillations, and v_n are the Fourier amplitudes of the expansion

$$v = \sum_{n=-\infty}^{\infty} v_{ne}^{inkx}; \qquad v_{-n} = v_n^*. \tag{5}$$

The equations of motion (2) are then replaced by

$$\dot{v}_n = ikn\,\partial H_0/\partial v_{-n}; \qquad \dot{v}_{-n} = -ikn\,\partial H_0/\partial v_n. \tag{6}$$

Since the wave solution has the form $v = v(x-ut)$, where u is the wave velocity, in formula (5) we represent the expansion coefficients in the form

$$v_n = a_{ne}^{-inkut}; \qquad a_{-n} = a_n^*, \tag{7}$$

where a_n do not depend on t and are known functions of u. For a nonlinear periodic wave there exists a one-to-one relation $H_0 = H_0(k,u)$, and henceforth we shall regard the quantities a_n and u as functions of H_0 and k.

We now represent the Hamiltonian (3) in the form

$$H(\Lambda) = H(\overline{\Lambda}) + H_1; \tag{8}$$

$$H(\overline{\Lambda}) = \frac{1}{2}\overline{\Lambda}\int dq\, q^2 v(q) v(-q) - \frac{1}{6}\int dq_1 dq_2 dq_3\, v(q_1) v(q_2) v(q_3) \delta(q_1+q_2+q_3);$$

$$H_1 = -\frac{1}{2}\int dq_1 dq_2 dq_3\, q_1 q_2 v(q_1) v(q_2) [\Lambda(q_3) - \overline{\Lambda}(q_3)\delta(q_3)]\, \delta(q_1+q_2+q_3). \tag{9}$$

It is not difficult to see that $\overline{\Lambda} = \frac{1}{2\pi}\int dx\,\Lambda(x)$. Below it will become clear that in the case $kR \gg 1$, where R is the characteristic scale of variation of Λ: $R \sim \Lambda/(d\Lambda/dx)$, the quantity $H_1 \ll H(\overline{\Lambda})$. This makes it possible to seek the solution in a form analogous to (7), (9):

$$v = \sum_n a_n(t) e^{in(kx-\vartheta)}, \tag{10}$$

where

$$\dot{a}_n/a_n \sim O(H_1); \qquad \dot{u}/u \sim O(H_1); \qquad \dot{\vartheta} = ku(t) + O(H_1).$$

Substitution of (10) into (9) gives

$$H(\overline{\Lambda}) = \frac{1}{2}\overline{\Lambda}\sum_n (kn)^2 v_n v_{-n} - \frac{1}{6}\sum_{n_1+n_2+n_3=0} v_{n_1} v_{n_2} v_{n_3} =$$

$$= \frac{1}{2}\overline{\Lambda}\sum_n (kn)^2 a_n a_{-n} - \frac{1}{6}\sum_{n_1+n_2+n_3=0} a_{n_1} a_{n_2} a_{n_3}; \tag{11}$$

$$H_1 = -\frac{1}{2}\sum_{n_1+n_2+n_3=0} (kn_1)(kn_2) v_{n_1} v_{n_2} [\Lambda_{n_3} - \Lambda_0]; \tag{12}$$

$$\Lambda_n = \frac{1}{2\pi}\int dx\,\Lambda(x) e^{-inkx}; \qquad \overline{\Lambda} = \Lambda_0; \qquad \Lambda_{-n} = \Lambda_n^*. \tag{13}$$

From formula (11) it follows that the functional relation between the quantities $H(\overline{\Lambda})$, u, and k is the same as in a homogeneous medium with $\Lambda(x) = \overline{\Lambda} = \Lambda_0$. From (12) it follows that only terms Λ_{n_3}, $n_3 \geq 1$, enter the expression for H_1. From (13) and inequality (11) it follows that all Λ_n with $n \geq 1$ are exponentially small and have the order

$$\Lambda_n \propto \exp(-nkR), \tag{14}$$

and, for a given $\Lambda(x)$, all Λ_n are in principle computable and are assumed to be known.

Estimate (14) shows that, up to terms of the next order in (12), it is sufficient to retain only the part containing Λ_1:

$$H_1 = -\frac{1}{2}\sum_n \{(kn)[k(1+n)] v_n v_{-1-n} \Lambda_1 +$$

$$+(kn)[k(n-1)]v_n v_{1-n}\Lambda_{-1}\} + O(\Lambda_1^2). \tag{15}$$

We now compute the change of $H(\overline{\Lambda})$ with time

$$\frac{dH(\overline{\Lambda})}{dt} = \sum_n \left\{ \frac{\partial H(\overline{\Lambda})}{\partial v_n}\dot{v}_n + \frac{\partial H(\overline{\Lambda})}{\partial v_{-n}}\dot{v}_{-n} \right\}.$$

For the quantities $\dot{v}_{\pm n}$ we use the exact expressions (6), in which H_0 must be replaced by $H(\Lambda)$, defined by formulas (8), (11), and (15). This gives

$$\dot{H}(\overline{\Lambda}) = \sum_n ikn \left(\frac{\partial H(\overline{\Lambda})}{\partial v_n}\frac{\partial H_1}{\partial v_{-n}} - \frac{\partial H(\overline{\Lambda})}{\partial v_{-n}}\frac{\partial H_1}{\partial v_n} \right). \tag{16}$$

Taking into account that the right-hand side in (16) is of order of smallness Λ_1, we write, using (6) and (10):

$$\frac{\partial H(\overline{\Lambda})}{\partial v_{\pm n}} = \mp \frac{1}{ikn}\dot{v}_{\mp n} + O(\Lambda_1) = \mp iuv_{\mp n} + O(\Lambda_1). \tag{17}$$

Substitution of (15), (12), and (17) into (16) leads to the equation

$$\dot{H}(\overline{\Lambda}) = i\sum_n \dot{\vartheta}(kn)^2\{(1-n)a_{-n}a_{n-1}\Lambda_1 e^{i\vartheta} -$$

$$-(1+n)a_{-n}a_{n+1}\Lambda_{-1}e^{-i\vartheta}\} + O(\Lambda_1^2); \tag{18}$$

$$\dot{\vartheta} = ku + O(\Lambda_1).$$

In addition, with the same accuracy one should regard $a_n = a_n(k, H(\overline{\Lambda}))$; $u = u(k, H(\overline{\Lambda}))$, and the variation of $H(\overline{\Lambda})$ in the right-hand side should be neglected. The obtained expression (18) makes it possible to compute the exponentially small change of $H(\overline{\Lambda})$

$$\Delta H(\overline{\Lambda}) = \lim_{T\to\infty}\int_{-T}^{T}\dot{H}(\overline{\Lambda})\,dt =$$

$$= \lim_{T\to\infty} i\sum_n (kn)^2\{(1-n)a_{-n}a_{n-1}\Lambda_1(e^{i\vartheta_+} - e^{i\vartheta_-}) -$$

$$-(1+n)a_{-n}a_{n+1}\Lambda_{-1}(e^{-i\vartheta_+} - e^{-i\vartheta_-})\} + O(\Lambda_1^2), \tag{19}$$

where $\vartheta_\pm = \vartheta(\pm T)$. Before passing to the limit $T \to \infty$, we carry out auxiliary estimates of the factors containing the phases $\vartheta_\pm$:

$$e^{i\vartheta_+} - e^{i\vartheta_-} = \exp\left\{ i \int_{-T}^{T} \omega(t)\, dt \right\} (1 + O(\Lambda_1 T)); \qquad \omega(t) = ku(t), \tag{20}$$

where in what follows we regard T as sufficiently large, but*

$$\omega T \Lambda_1/\Lambda_0 \ll 1. \tag{21}$$

With the same accuracy one may obtain

$$\int_{-T}^{T} \omega(t)\, dt = 2T\omega(0) + O\left(\frac{d\omega}{dH}\,\Delta H \cdot T\right). \tag{22}$$

Let now $T = m\pi/\omega(0)$, where m is an integer, and pass to the limit $m \to \infty$. After substituting (22) into (20) we obtain, as $m \to \infty$,

$$e^{i\vartheta_+} - e^{i\vartheta_-} = 1 + O\left(\frac{\Lambda_1}{\Lambda_0}\,\frac{d\omega/dH}{\omega/H}\,\omega T\right). \tag{23}$$

Taking the above into account and substituting (23) into (19), we obtain the expression for the change of energy in a more convenient form:

$$\Delta H(\overline{\Lambda}) = 2k^2|\Lambda_1| \sum_n n^2(n-1)|a_n| \cdot |a_{n-1}| \sin(\varphi - \theta) + O(kRe^{-2kR});$$

$$\Lambda_1 = |\Lambda_1|e^{i\varphi}; \qquad a_p = |a_p|e^{i\theta_p} = |a_p|e^{ip\theta}. \tag{24}$$

By means of formula (24) one finds the changes of the other quantities characterizing the nonlinear periodic wave:

$$\Delta u = -\frac{du}{dH(\overline{\Lambda})}\Delta H(\overline{\Lambda}); \qquad \Delta a_n = -\frac{da_n}{dH(\overline{\Lambda})}\Delta H(\overline{\Lambda}),$$

which are also exponentially small.

* Restriction (21) can in fact be removed if one takes into account that the effective change of the wave energy $H(\overline{\Lambda})$ occurs on an interval of length $\sim R$, i.e., over a time $T_0 \sim R/u$. Therefore instead of (21) we have the inequality $(\Lambda_1/\Lambda_0)\omega R/u \sim kR\exp(-kR) \ll 1$, which is automatically satisfied for $kR \gg 1$.

To make sure that the result obtained in (24) is directly related to the adiabatic invariant of the wave, let us perform some transformations. Rewrite (11) in the form:

$$H(\overline{\Lambda}) = -\frac{1}{2}\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} dx\,(\overline{\Lambda}v_x^2 + \tfrac{1}{3}v^3) = -\frac{1}{2}\overline{\Lambda}kI - \frac{1}{6}C;$$

$$I = \frac{1}{2\pi}\oint v_x^2\,dx = \frac{1}{2\pi}\oint v'\,dv; \qquad C = \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} v^3\,dx, \tag{25}$$

where the integration in I is carried out over the period of the wave and the prime denotes differentiation with respect to the argument $x - ut$. The quantity I is the adiabatic invariant of the wave. Since C is a function only of k and $H(\overline{\Lambda})$, it follows from (25) that

$$\Delta I = -\frac{2}{k\overline{\Lambda}}\left(1 + \frac{1}{6}\frac{\partial C}{\partial H(\overline{\Lambda})}\right)\Delta H(\overline{\Lambda}).$$

This expression is the final result and determines the change of the adiabatic invariant of the wave in a weakly inhomogeneous medium. The quantity $\partial C/\partial H(\overline{\Lambda})$ is of order unity, while $\Delta H(\overline{\Lambda})$, according to (24), is proportional to Λ_1; consequently, ΔI is exponentially small.

Let us consider some applications of the results obtained.

1. Let the periodic solution of equation (1) have a very large wavelength ($k \to 0$, but $kR \gg 1$). It is convenient to introduce a number N characterizing the effective number of harmonics in the expansion (5). As is known, in this case

$$a_n \approx 3u/N \quad (n \lesssim N); \qquad a_n \sim \exp(-n/N) \quad (n \geq N).$$

As $k \to 0$, the number of harmonics $N \gg 1$, and $N = \frac{1}{k}\sqrt{3u/\overline{\Lambda}}$. Formula (24) is then simplified:

$$\Delta H(\overline{\Lambda}) \approx 2^{7/2}u^3\frac{|\Lambda_1|}{\overline{\Lambda}}\sin(\varphi - \theta). \tag{26}$$

A special feature of expression (26) is the absence of dependence on k. Since $H(\overline{\Lambda}) \propto k$, this means that the inequality $\Delta H(\overline{\Lambda})/H(\overline{\Lambda}) \ll 1$, necessary for the applicability of perturbation theory, as $k \to 0$ may cease to hold even before the quantity kR becomes of order unity.

2. Consider the case of a perturbation in the form of a moving profile: $\Lambda = \Lambda(x - u_\Lambda t)$. Making in (1) the change of variables $y = x - u_\Lambda t,\ t = t$, we obtain

$$v_t - u_\Lambda v_y + v v_y + \Lambda(y) v_{yyy} = 0, \tag{27}$$

and the problem reduces to the one already considered, with some changes of notation. In particular, instead of the solution $v(x - ut)$ for equation (1), one should consider $v = v(y - (u - u_\Lambda)t)$ for equation (27).

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Note: Figure translations are in progress. See original paper for figures.