ON BANACH ANALYTIC SPACES AND THE MODULI SPACE OF HOLOMORPHIC FIBRATIONS

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 517.53

MATHEMATICS

I. F. DONIN

ON BANACH ANALYTIC SPACES AND THE MODULI SPACE OF HOLOMORPHIC FIBRATIONS

(Presented by Academician P. S. Aleksandrov, 22 V 1970)

The main difficulty in the local study of Banach manifolds arises from the fact that not every closed subspace in a Banach linear space is complemented, i.e., has a complementary subspace in the topological sense. This makes it possible to carry over the usual "finite-dimensional" proofs to the infinite-dimensional case.

The purpose of the present work is to show that in many cases the question of complementability can be bypassed. This is achieved with the aid of Theorems 1 and 2. As an application of the results obtained, the existence of a complete moduli space for holomorphic fibrations with arbitrary structure group over a compact complex space is established.

1°. Most of the definitions concerning Banach analytic spaces are contained in the work (^{1}). We shall call an analytic set finite-dimensional if it is embedded in a neighborhood of a finite-dimensional numerical space.

Definition 1. A Banach analytic space X will be called **complete** at a point $x \in X$ if there exists a neighborhood V of the point x, isomorphic to the analytic set $\alpha^{-1}(o)$, where $\alpha : F \to G$ is an analytic mapping of Banach manifolds, $o \in G$, and the differential $\dot{\alpha}$ at the point x has closed image.

Let X be a Banach analytic space, $o \in X$, and let H be a finite-dimensional subspace of the tangent space to X at the point o.

Definition 2. A finite-dimensional analytic subset in a neighborhood of the point o is called **maximal relative to** H if it has H as its tangent space at the point o and is not contained in any finite-dimensional analytic subset possessing this property.

Theorem 1. *Let X be a Banach analytic space, complete at the point $o \in X$, and let H be a finite-dimensional subspace of the tangent space to X at the point o. Then in some neighborhood of the point o there exists an analytic subset maximal relative to H.*

For the proof, one may assume without loss of generality that $X = \alpha^{-1}(o')$, where $\alpha : F \to G$ is an analytic mapping of Banach manifolds, $o' \in G$, and the differential $\dot{\alpha}$ at the point o has closed image. One may further assume that F and G are neighborhoods of zero, respectively, in the vector Banach spaces $\mathbf{F}$ and $\mathbf{G}$.

The proof of the theorem is carried out as follows. We shall seek an analytic mapping $\varphi : V \to F$, V a neighborhood of zero in H, in the form of a convergent series $\varphi = \varphi_1 + \dots + \varphi_n + \dots$, where φ_n is an n-linear form; moreover, φ_1 is the identity mapping of H onto itself. The desired maximal analytic set will have the form $A = [\alpha \cdot \varphi]^{-1}(o')$.

The composition φ with α must expand into a series of the form

$$\alpha \cdot \varphi = \alpha_1(\varphi_1) + [\alpha_1(\varphi_2) + \alpha_2(\varphi_1, \varphi_1)] + \cdots,$$

where α_n is an n-linear continuous form from F to G in the expansion into a convergent series of the mapping α.

Thus, φ_1 is known to us, $\alpha_2(\varphi_1, \varphi_1)$ is a bilinear form from H to G, which may be regarded as a linear mapping

$$H \otimes H \xrightarrow{\lambda_2} G.$$

Put $L = \lambda_2^{-1}(\operatorname{Im} \alpha_1)$ and $\psi = \lambda_2 \pi$, where π is a linear projection of $H \otimes H$ onto L. The form ψ_2 can be lifted to a form $\widetilde{\psi}_2$ from H to F in such a way that $\alpha_1(\widetilde{\psi}_2) = \psi_2$. As φ_2 we take $-\widetilde{\psi}_2$.

Analogously one constructs the form φ_n, if the forms $\varphi_1, \dots, \varphi_{n-1}$ are known. By the closedness of the image of α_1, the standard estimates are easily carried out and the convergence of the series $\varphi = \varphi_1 + \varphi_2 + \cdots$ is proved. The fact that A is maximal under this construction is proved algebraically.

Remark 1. The preceding arguments make it possible to prove the existence of a maximal analytic set containing, in advance, a given analytic subset with tangent space at the point o contained in H.

Theorem 2. *Let X be a Banach analytic space, $o \in X$, let A be a maximal analytic subset in a neighborhood of the point o, let D be a Banach manifold with a fixed point $o' \in D$, and let $\alpha : A \times D \to X$ be an analytic mapping inducing an embedding on $A \times o'$, and such that the image of the differential $\dot{\alpha}$ at the point (o, o') coincides with the tangent space to X at the point o.*

Then for any analytic mapping $f : B \to X$, where B is a finite-dimensional analytic set with a fixed point $b \in B$, there exists a neighborhood of the point b and an analytic mapping g of this neighborhood into $A \times D$ such that $\alpha g = f$, $g(b) = (o, o')$.

The proof of this theorem is similar to the proof of Theorem 1: the mapping g is sought in the form of a convergent power series.

Corollary 1. *Let X be a Banach analytic space with finite-dimensional tangent space T at the point $o \in X$, and complete at the point o. Then in a neighborhood of the point o there exists a unique finite-dimensional analytic subset A, containing all other finite-dimensional analytic subsets in X.*

As A one must take the analytic set maximal with respect to T. Its uniqueness follows from Theorem 2, if one takes D to be a single point. In fact, even any analytic mapping of a finite-dimensional analytic set into X may be regarded as a mapping into A.

If X were isomorphic to the analytic set $\alpha^{-1}(o')$, where $\alpha : F \to G$ is an analytic mapping of Banach manifolds, and the space $\operatorname{Im} \dot{\alpha}$ were complemented, Corollary 1 would be trivial, since X itself would be finite-dimensional.

Corollary 2. *Let $f : E \to F$ be a linear homomorphism of Banach spaces, i.e. $\operatorname{Im} f$ is closed. Suppose that $\varphi : B \to \operatorname{Im} f$ is an analytic mapping of a finite-dimensional analytic set B into $\operatorname{Im} f$, regarded as a submanifold in F. Then there exists an analytic mapping $\widetilde{\varphi} : B \to E$ such that $f \cdot \widetilde{\varphi} = \varphi$.*

This corollary shows especially clearly how, with the help of Theorem 2, the difficulty connected with complementability is bypassed. If $\operatorname{Ker} f$ and $\operatorname{Im} f$ were complemented subspaces, then, as in the finite-dimensional case, the assertion would be trivial.

2°. We now apply the preceding considerations to deformations of holomorphic bundles.

Let X be a compact complex space, G a complex Lie group, and P a holomorphic bundle with base X and structure group G.

We shall use the notions of [2]. Let $U = \{U_i\}$ and $V = \{V_i\}$ be finite open coverings of the space X by holomorphically complete coordinate neighborhoods, properly inscribed in one another, i.e. $\overline{V}_i \subset U_i$ for all i. Denote by F_0 the infinite-dimensional Lie group of zero-dimensional holomorphic noncommutative cochains, defined on the covering of the space X by the compact sets $\overline{V} = \{\overline{V}_i\}$ with values in the group G, and by F_1 the infinite-dimensional manifold of the corresponding one-dimensional cochains. In F_1 the equations $\{f_{ij} f_{jk} f_{ki} = 1\}$ determine an infinite-dimensional analytic subspace Z_1 of one-dimensional noncommutative cocycles. The group F_0 acts analytically on Z_1 as follows: if $\alpha = \{t_i\} \in F_0$ and $\beta = \{f_{ij}\} \in Z_1$, then $\alpha \circ \beta = \{t_i f_{ij} t_j^{-1}\} \in Z_1$.

Let the bundle P be given by the cocycle $\omega \in Z_1$. Then the tangent space to Z_1 at the point ω can be naturally identified with the space of one-dimensional cocycles with values in $\operatorname{Ad} P$ in the covering $\overline{V}$, while the image of the differential at the identity of the mapping $\alpha \mapsto \alpha \circ \omega$, where $\alpha \in F_0$, is identified with the space of the corresponding one-dimensional coboundaries. Thus the complement to the coboundaries in the tangent space to Z_1 at the point ω is identified with the cohomology space $H^1(X, \operatorname{Ad} P)$, whose dimension, as is known, is finite.

Formally, in order to prove the existence of a complete family of deformations of the bundle P, we could apply the process described in item 1°. Thus, with the aid of Theorem 1, we would construct an analytic subset A in Z_1, maximal with respect to $H^1(X, \operatorname{Ad} P)$, and with the aid of Theorem 2 establish the completeness of the family corresponding to this subset. However, here we are not dealing with Banach manifolds; therefore, in order to prove the convergence of this process (the convergence of the series φ in Theorem 1), it is necessary, for each two-dimensional coboundary $\{f_{ijk}\}$, to be able to construct a cochain $\{f_{ij}\}$ resolving it, $f_{ij} + f_{jk} - f_{ik} = f_{ijk}$, with a sufficiently good estimate.

The following assertion holds:

$*$ Let S be a coherent sheaf on X, let $V' = \{V'_i\}$ be a covering of the space X such that $U \supset V' \supset V$, and let ε be a positive real number such that the ε-neighborhood V'_ε of the covering V' is a covering inscribed in U. Then, for any n-dimensional coboundary ξ_n, defined in the covering V'_ε with values in S, there exists an $(n-1)$-dimensional cochain η_{n-1} resolving it, for which an estimate of the form

$$|\eta_{n-1}|_{V'} \le C \, \frac{1}{\varepsilon^\nu} \, |\xi_n|_{V'_\varepsilon}, \tag{*}$$

holds, where the constants C and ν depend only on n, but do not depend on the covering V' or on ε.

We shall not go into the definition of the norms and the ε-neighborhood in detail because of their cumbersomeness, but we shall indicate the idea of the proof of the assertion. By means of an analogue of Leray's theorem (with estimates taken into account), the assertion reduces to the local situation, where the following holds.

Proposition. *Let S be a coherent sheaf defined in a neighborhood of a convex bounded open set D of a complex number space, let D_ε be the ε-neighborhood of the set D, and let V and V_ε be, respectively, convex coverings of the set D and of its ε-neighborhood. Then, for every n-cocycle ξ_n with values in S in the covering V_ε, there exists an $(n-1)$-cochain η_{n-1} resolving ξ_n, such that an estimate of the form $(*)$ is satisfied, where the constants C and ν depend only on the diameters of the sets D and V, on the sheaf S, and on n.*

If S is a free sheaf, then the proposition is easily obtained with the aid of Hörmander' s Theorem 2.2.3 on L_2-cohomology from [3]. If

if S is an arbitrary sheaf, then the proof proceeds by induction on the length of the resolution of the sheaf S, with the aid of the following lemma.

Lemma. Let $O^p \to O^q$ be a homomorphism of free sheaves over a neighborhood of the domain D. Then, for every section ξ of the sheaf O^q over D_ε belonging to Im f, there exists a section η over D of the sheaf O^p such that the estimate $(*)$ holds, where C and ν depend only on the diameter of the set D and on f.

This lemma is proved by means of the well-known Łojasiewicz inequality (see (^{4}), p. 73).

Taking assertion $*$ into account and applying the methods of item 1°, we obtain the following theorem.

Theorem 3. Let X be a compact complex space, G a complex Lie group, and P a holomorphic fiber bundle over X with structure group G. Then P has a complete finite-dimensional family of deformations with tangent space of dimension $\dim H^1(X, \operatorname{Ad} P)$.

Remarks. 1) It is easy to show that the complete family is defined uniquely up to isomorphism. 2) It can be shown that if $H^2(X, \operatorname{Ad} P) = 0$, then the complete family will be a manifold.

Received
15 V 1970

REFERENCES

[1] A. Donady, Ann. Inst. Fourier, Grenoble, **16**, 1, 1 (1966).
[2] . . Donin, Mat. sborn., **77** (119), 4, 602 (1968).
[3] L. Hörmander, Sborn. per. Matematika, **10**, 2, 59 (1966).
[4] B. Malgrange, *Ideals of Differentiable Functions.* Moscow, 1968.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.