FREQUENCY CONDITIONS FOR ABSOLUTE STOCHASTIC STABILITY OF AUTOMATIC CONTROL SYSTEMS WITH RANDOM EXTERNAL ACTION

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Abstract

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MATHEMATICS

M. V. LEVIT

FREQUENCY CONDITIONS FOR ABSOLUTE STOCHASTIC STABILITY OF AUTOMATIC CONTROL SYSTEMS WITH RANDOM EXTERNAL ACTION

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1°. Consider the integral equation describing a control system with n nonlinear blocks $\varphi_j = \varphi_j(t, \sigma)$:

$$\sigma_t = \alpha(t) + \int_0^t \Omega(t-\tau)\varphi_\tau \, d\tau, \qquad \varphi_t = \varphi(t, \sigma_t), \qquad t \geqslant 0. \tag{1}$$

Here $\alpha(t)$, σ_t are vector random processes (of order m), $\varphi(t, \sigma) = \|\varphi_j(t, \sigma)\|$ is a vector function of order n, continuous in the aggregate of the variables t and σ ($t \geqslant 0$, $-\infty < \sigma_i < +\infty$), and $\Omega(t)$ is a matrix function of order $m \times n$. We shall assume that $\Omega(t)$ is summable on the half-axis $(|\Omega(t)| \in L[0, +\infty))^*$, and that the measurable process $(^1)$ $\alpha(t)$ is, with probability 1 (w.p. 1), square-summable on every finite interval ($|\alpha(t)| \in L_2[0, T]$ w.p. 1 for any $T > 0$).

A solution of system (1) on the interval $[0, T]$ will mean a pair of measurable processes $y_t = (\sigma_t, \varphi_t)$, square-summable w.p. 1 on $[0, T']$, $T' < T$, and satisfying (1) almost everywhere on $[0, T]$, w.p. 1.

By analogy with $(^2)$, we shall say that the strengthened local existence theorem holds for system (1) if the following are true: (A) system (1) has a solution on some interval $[0, T_0]$; (B) if this solution is square-summable on $[0, T_0]$ w.p. 1, then it is extendable to some wider interval $[0, T_1]$, $T_1 > T_0$.

The strengthened local existence theorem holds for a fairly broad class of systems (1). It holds, for example, in the case when $\varphi(t, \sigma)$ satisfies a Lipschitz condition with respect to the argument σ with a constant independent of t, and

$$\int_0^T E|\alpha(t)|^2 \, dt < +\infty$$

for any $T > 0$**. The proof is carried out by means of the contraction mapping principle for the operator

$$A(\sigma_t) = \int_0^T \Omega(t-\tau)\varphi(\tau, \sigma_\tau)\, d\tau + \alpha(t)$$

in the space $\mathfrak{E}[0, T]$ of measurable random processes σ_t $(0 \leqslant t \leqslant T)$ such that

$$\int_0^T E|\sigma_t|^2\, dt < +\infty^{***}.$$

Denote by $\mathfrak{A}$ the set of external actions $\alpha(t)$ for which system (1) has a solution on the half-axis $[0, +\infty)$. System (1) defines a mapping U of the set $\mathfrak{A}$ onto the set Y of corresponding solutions $y_t = (\sigma_t, \varphi_t)$. Suppose that nonnegative functions $r(\alpha)$ and $\rho(y)$ are given on $\mathfrak{A}$ and Y, respectively, possibly taking the value $+\infty$, with $r(0) = 0$, $\rho(0) = 0$. We shall say that system (1) is stochastically stable

with respect to the functions $r(\alpha)$ and $\rho(y)$, if the following hold: a) from the condition $r(\alpha) < +\infty$ it follows that $\rho(y) < +\infty$; b) from the condition $r(\alpha) \to 0$ it follows that $\rho(y) \to 0$.

We shall call the class of systems (1) absolutely stochastically stable with respect to $r(\alpha)$ and $\rho(y)$ if every system in the class is stochastically stable with respect to $r(\alpha)$ and $\rho(y)$.*

In Theorems 1-4 below, sufficient conditions are given for the existence of solutions of system (1) on the half-axis $[0, +\infty)$ and for the fulfillment, for any $T > 0$, of one of the inequalities

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* Here and below,

$$|A| = \sqrt{\sum_{j,h} |a_{jh}|^2}$$

$(A = \|a_{jh}\|)$ is a rectangular matrix.

** Here and below, the symbol E denotes mathematical expectation.

*** Obviously, at the same time existence and uniqueness of the solution in $\mathfrak{E}[0, T]$ will be proved. Below we also use the space $\mathfrak{E}[0, +\infty)$ of measurable random processes σ_t $(0 \leqslant t < +\infty)$ such that

$$\int_0^{+\infty} E|\sigma_t|^2\, dt < +\infty.$$

$$\int_0^T E|y_t|^2\,dt \le M^2 \int_0^T E|\alpha(t)|^2\,dt + N\gamma + Q, \tag{2}$$

$$\int_0^T E(|y_t|^2 + |\dot\sigma_t|^2)\,dt \le M^2 \int_0^T E(|\alpha(t)|^2 + |\dot\alpha(t)|^2)\,dt + N\gamma + Q, \tag{3}$$

in which the nonnegative constants M, N, γ, and Q do not depend on T. Using (2) and (3), it is easy to establish the stochastic stability of system (1) with various $r(\alpha)$ and $\rho(y)$. Namely:

Proposition 1. Suppose that for the solutions of system (1) on the half-axis $[0, +\infty)$, (2) is satisfied. Then: (I) system (1) is stochastically stable with respect to the seminorms

$$r_1(\alpha) = \lim_{T\to\infty} \left(\frac{1}{T}\int_0^T E|\alpha(t)|^2\,dt\right)^{1/2}, \qquad \rho_1(y) = \lim_{T\to\infty} \left(\frac{1}{T}\int_0^T E|y_t|^2\,dt\right)^{1/2};$$

(II) if $\gamma = Q = 0$, then the system is stochastically stable with respect to the norms

$$r_2(\alpha) = \sup_T \left(\frac{1}{T}\int_0^T E|\alpha(t)|^2\,dt\right)^{1/2}, \qquad \rho_2(y) = \sup_T \left(\frac{1}{T}\int_0^T E|y_t|^2\,dt\right)^{1/2}$$

and the norms

$$r_3(\alpha) = \left(\int_0^{+\infty} E|\alpha(t)|^2\right)^{1/2}, \qquad \rho_3(y) = \left(\int_0^{+\infty} E|y_t|^2\,dt\right)^{1/2}.$$

Proposition 2. Suppose that $\Omega(t)$ and $\alpha(t)$ (see Section 1) are absolutely continuous; then, from Section 1, the process σ_t is absolutely continuous. If (3) is satisfied, then assertions (I), (II) of Proposition 1 are valid, in which $r_j(\alpha)$ and $\rho_j(y)$ $(j = 1, 2, 3)$ are replaced respectively by $r_j(\tilde\alpha)$ and $\rho_j(\tilde y)$, where

$$\tilde\alpha(t) = \sqrt{|\alpha(t)|^2 + |\dot\alpha(t)|^2}, \qquad \tilde y(t) = \sqrt{|y_t|^2 + |\dot\sigma_t|^2}.$$

Moreover, if $\alpha(t) \in \mathfrak{C}[0, +\infty)$, $\dot\alpha(t) \in \mathfrak{C}[0, +\infty)$, then $E|\sigma_t|^2 \to 0$ as $t \to \infty$.

2°. Quite often, the specific properties of the nonlinear blocks of system (1) make it possible to assert that the solutions of the system in Section 1 satisfy, on the interval of existence, the relation

$$\int_0^t F(\varphi_\tau, \sigma_\tau, \dot{\sigma}_\tau)\, d\tau \geq -\gamma, \tag{4}$$

where F is a real quadratic form of its arguments and γ is a nonnegative constant. Diverse examples of such relations are given in (^{4}), and the method of using them to study the stability of deterministic systems (1) is set forth in (^{2}). Following (^{2}), we extend each real quadratic form $F(\varphi, \sigma, \dot{\sigma})$ (the vectors $\varphi, \sigma, \dot{\sigma}$ have orders n, m, m) to complex values of the arguments, preserving Hermitian symmetry, and define the function $F(p, \bar{\varphi}) = F(\bar{\varphi}, -\chi(p)\bar{\varphi}, -p\chi(p)\bar{\varphi})$. Here $p = i\omega$, $\omega \in (-\infty, +\infty)$, $\bar{\varphi}$ is a complex n-vector, and $-\chi(p)$ is the Laplace transform of the function $\Omega(t)$.

* For the first time, as far as the author knows, a similar definition of absolute stochastic stability was given in (^{3}). There, under certain additional assumptions concerning $\alpha(t)$, theorems were proved that are special cases of Theorems 1 and 3 of the present paper.

Theorem 1. Suppose that for system (1) the strengthened local existence theorem is satisfied. Let the solutions of (1) with p. 1 satisfy relation (4), in which $F = F(\varphi, \sigma)$ does not depend on $\dot{\sigma}$. Let the following hold: (A_1) $F(0, \sigma) \geqslant 0$ for all real σ; (B_1) $\tilde{F}(p, \tilde{\varphi}) < 0$ for all complex $\tilde{\varphi}$ and $p = i\omega$, $-\infty \leqslant \omega \leqslant +\infty$. Then the solutions of system (1) are continuable to the half-axis $[0, +\infty)$, and (2) is valid for $Q = 0$.

Theorem 2. Suppose that for system (1) all the conditions of Theorem 1 are satisfied, except for (A_1). Let it be known that $|\varphi_t| \leqslant \text{const}$ with p. 1, and that $\Omega(t)$ satisfies the relation

$$\lim_{T\to\infty} \int_0^T \int_{T-\tau}^{\infty} |\Omega(t)|\, dt\, d\tau < +\infty * .$$

Then the solutions of system (1) are continuable to the half-axis $[0, +\infty)$, and (2) is valid.

Theorem 3. Suppose that in system (1) $\Omega(t)$ and $\alpha(t)$ (with p. 1) are absolutely continuous, moreover $|\Omega(t)| \in L[0, +\infty)$, and $|\dot{\alpha}(t)| \in L_2[0, T]$, $T > 0$ with p. 1. Let the strengthened local existence theorem be satisfied. Let the solutions of system (1) with p. 1 satisfy relation (4). Let the following hold: (A_2) $F(0, \sigma, \dot{\sigma}) \geqslant 0$ for all real σ and $\dot{\sigma}$; (B_2) $\tilde{F}(p, \tilde{\varphi}) < 0$ for all complex $\tilde{\varphi}$ and $p = i\omega$, $-\infty \leqslant \omega \leqslant +\infty$. Then the solutions of system (1) are continuable to the half-axis $[0, +\infty)$, and (3) is valid for $Q = 0$.

Theorem 4. Suppose that for system (1) all the conditions of Theorem 3 are satisfied, except for (A_2). Let it be known that $|\varphi_t| \leqslant \text{const}$ with p. 1 and

$$\lim_{T\to\infty}\int_0^T\int_{T-\tau}^{\infty}(|\Omega(t)|+|\dot{\Omega}(t)|)\,dt\,d\tau<+\infty.$$

Then the solutions of (1) are continuable to the half-axis $[0,+\infty)$, and (3) is valid.

3°. We shall briefly indicate the main points in the proofs of the formulated assertions.

Proposition 1 is obvious. In Proposition 2 only the last assertion requires proof. Note that from the conditions $\alpha(t)\in\mathfrak{C}[0,+\infty)$, $\dot{\alpha}(t)\in\mathfrak{C}[0,+\infty)$ and inequality (3) it follows that $\sigma_t\in\mathfrak{C}[0,+\infty)$, $\dot{\sigma}_t\in\mathfrak{C}[0,+\infty)$. Then from

$$E(|\sigma_t|^2-|\sigma_0|^2)=2\int_0^t E(\sigma_\tau,\dot{\sigma}_\tau)\,d\tau$$

we obtain that there exists a finite limit $\lim_{t\to\infty}E|\sigma_t|^2$, and this limit is equal to zero.

Theorems 1–4 are a probabilistic analogue of the results obtained in (^{2}) for deterministic systems (1). Therefore the proof of many points is based on the same ideas and is carried out analogously.

Proof of Theorems 1 and 2. As in (^{2}), if the transfer function $\chi(i\omega)$ and the quadratic form $F(\varphi,\sigma)$ satisfy condition (B_1), then there exists a constant $\delta>0$ such that

$$\tilde{F}(i\omega,\tilde{\varphi})+2\delta(1+|\chi(i\omega)|^2)|\tilde{\varphi}|^2\leqslant 0.$$

Consider the random variable

$$I(T)=\int_0^T[F(\varphi_\tau,\xi_\tau)+2\delta(|\varphi_\tau|^2+|\xi_\tau|^2)]\,d\tau.$$

Here T belongs to the interval of existence of the solution $y_t=(\sigma_t,\varphi_t)$ of system (1) **,** $\xi_1=\sigma_t-\alpha(t)=\Omega(t)*\varphi_t$ *.

Put $\varphi_t^T=\varphi_t$ for $0<t\leqslant T$, $\varphi_t^T=0$ for $t>T$ or $t<0$, $\xi_t^T=\Omega(t)*\varphi_t^T$. Then

$$I(T)=J(T)-\int_T^{+\infty}(F(0,\xi_\tau^T)+2\delta|\xi_\tau^T|^2)\,d\tau,\quad\text{where } J(T)=$$

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* Under the limit sign there is a function nondecreasing in T and, consequently, the limit exists.

** The existence of such an interval is guaranteed by the strengthened local existence theorem.

*** The symbol $*$ denotes the convolution operation.

$$= \int_{-\infty}^{+\infty} [F(\varphi_\tau^T, \xi_\tau^T) + 2\delta(|\varphi_\tau^T|^2 + |\xi_\tau^T|^2)]\, d\tau.$$

Since $|\varphi_t^T| \in L(-\infty,+\infty) \cap L_2(-\infty,+\infty)$ a.s., it follows that $|\xi_t^T| \in L(-\infty,+\infty) \cap L_2(-\infty,+\infty)$ a.s.; moreover, for their Fourier transforms the equality $\tilde{\xi}^T = -\chi(i\omega)\tilde{\varphi}^T$ holds a.s. Applying Parseval’ s equality to $J(T)$, we obtain that, by the choice of δ, $J(T) \leq 0$ a.s. Consequently,

$$I(T) \leq -\int_T^{+\infty} F(0, \xi_\tau^T)\, d\tau$$

a.s. Similarly to (2), it is easy to show the existence of constants $k_1 > 0$, $k_2 > 0$ such that, for any $\varepsilon > 0$, the inequality

$$-F(\varphi, \sigma - \alpha) - \varepsilon(|\varphi|^2 + |\sigma - \alpha|^2) \leq -F(\varphi, \sigma) + (k_1 + k_2/\varepsilon)|\alpha|^2$$

is valid. Take $\varepsilon = \delta$, $k = k_1 = k_2/\delta$. We obtain

$$\delta \int_0^T (|\varphi_\tau|^2 - |\xi_\tau|^2)\, d\tau = I(T) - \int_0^T [F(\varphi_\tau, \xi_\tau) + \delta(|\varphi_\tau|^2 + |\xi_\tau|^2)]\, d\tau$$
$$\leq -\int_T^{+\infty} F(0, \xi_\tau^T)\, d\tau - \int_0^T F(\varphi_\tau, \sigma_\tau)\, d\tau + k\int_0^T |\alpha(\tau)|^2 d\tau$$

a.s. Applying to the last inequality relation (4) for the form $F(\varphi, \sigma)$, we obtain that, under the conditions of Theorem 1 or 2, a.s. for all T from the interval of existence of the solution the relation

$$\delta \int_0^T (|\varphi_\tau|^2 + |\xi_\tau|^2)\, d\tau \leq \gamma - \int_T^{+\infty} F(0, \xi_\tau^T)\, d\tau + k\int_0^T |\alpha(\tau)|^2 d\tau.$$

Let the conditions of Theorem 2 be satisfied. Then

$$\left|\int_T^{+\infty} F(0, \xi_\tau^T)\, d\tau\right| \leq \text{const} \int_0^T \int_{T-\tau}^{+\infty} |\Omega(t)|\, dt\, d\tau \leq \text{const}.$$

Therefore there exist nonnegative constants M, N, and Q, independent of T, such that a.s.

$$\int_0^T (|\varphi_\tau|^2 + |\sigma_\tau^2|)\, d\tau \leq M^2 \int_0^T |\alpha(t)|^2\, dt + N\gamma + Q. \tag{5}$$

Under the conditions of Theorem 1, however, from (A_1) there follows inequality (5) with $Q = 0$. From (5) and the strengthened local existence theorem it follows that the solutions can be continued to the half-axis $[0, +\infty)$. Since φ_t, σ_t, and $\alpha(t)$ are measurable processes, estimate (2) follows from (5).

Proof of Theorems 3 and 4. Let the conditions of Theorem 3 be satisfied. By the strengthened local existence theorem, system (1) has a solution y_t on some interval. Differentiating the first equation of system (1), we obtain that the process $\hat{y}_t = (\hat{\sigma}_t, \varphi_t)$, where

$$\hat{\sigma}_t = \begin{pmatrix} \sigma_t \\ \dot{\sigma}_t \end{pmatrix},$$

will be a solution of a system of the form (1), in which the roles of $\alpha(t)$ and $\Omega(t)$ are played by

$$\hat{\alpha}(t) = \begin{pmatrix} \alpha(t) \\ \dot{\alpha}(t) \end{pmatrix} \quad \text{and} \quad \hat{\Omega}(t) = \begin{pmatrix} \Omega(t) \\ \dot{\Omega}(t) \end{pmatrix}.$$

For the system thus obtained all the conditions of Theorem 1 are valid. Then from (5), for some M and N, it follows a.s. that

$$\int_0^T (|\varphi_t|^2 + |\sigma_t|^2 + |\dot{\sigma}_t|^2)\, dt \leq M^2 \int_0^T (|\alpha(t)|^2 + |\dot{\alpha}(t)|^2)\, dt + N\gamma.$$

Hence, as before, the solutions y_t can be continued to the half-axis $[0, +\infty)$, and (3) is valid. Theorem 4 follows in the same way from Theorem 2.

Leningrad State University
named after A. A. Zhdanov

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Note: Figure translations are in progress. See original paper for figures.

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