ON INFINITELY DIVISIBLE LAWS HAVING ONLY INFINITELY DIVISIBLE COMPONENTS

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 519.81

MATHEMATICS

I. V. OSTROVSKII

ON INFINITELY DIVISIBLE LAWS HAVING ONLY INFINITELY DIVISIBLE COMPONENTS

(Presented by Academician Yu. V. Linnik, February 4, 1970)

One of the fundamental problems in the theory of decompositions of probability laws is the problem of describing the class I_0—the class of infinitely divisible laws having only infinitely divisible components. Studies by H. Cramér, P. Lévy, A. Ya. Khinchin, D. A. Raikov, and Yu. V. Linnik have been devoted to the study of this problem; their results are summarized in the monograph of Yu. V. Linnik ([1]). These results have recently been supplemented in papers ([2–9]). The problem has not been solved completely; however, a considerable number of various necessary or sufficient conditions for membership in I_0 have been obtained.

Yu. V. Linnik ([1]) (p. 79) introduced the notion of an α-component of a law, generalizing the notion of an ordinary component. Recall that a law F_1 is called an α-component of a law F if there exist laws $F_2, \dots, F_n$, positive numbers $\alpha_1, \dots, \alpha_n$, and a sequence $t_k \to 0$ such that, for $t = t_k$, we have

$$\varphi(t, F) = \{\varphi(t, F_1)\}^{\alpha_1} \cdots \{\varphi(t, F_n)\}^{\alpha_n}, \tag{1}$$

where $\varphi(t, F), \varphi(t, F_1), \dots, \varphi(t, F_n)$ are the characteristic functions of the laws $F, F_1, \dots, F_n$, respectively. Yu. V. Linnik drew the author' s attention to the problem of describing the class I_0^α—the class of infinitely divisible laws having only infinitely divisible α-components. Obviously, $I_0^\alpha \subseteq I_0$.

One of the results obtained by Yu. V. Linnik ([1]) (p. 242) may be regarded as a sufficient condition for membership in I_0^α. This result may be formulated as follows.

Let F be an infinitely divisible law,

$$\varphi(t, F) = \exp\left\{i\beta t + \int_{-\infty}^{\infty} \left(e^{itx} - 1 - \frac{itx}{1 + x^2}\right) \frac{1 + x^2}{x^2}\, dG(x)\right\} \tag{2}$$

be the representation of its characteristic function $\varphi(t, F)$ by the Lévy-Khinchin formula. In order that the law F belong to I_0^α, it is sufficient that the following conditions be fulfilled: a) the function $G(x)$ is a step function, and its discontinuity points are contained in a set of the form

$$\{0\} \cup \{\mu_{k1}\}_{k=-\infty}^{\infty} \cup \{\mu_{k2}\}_{k=-\infty}^{\infty},$$

where $\mu_{k1} > 0$, $\mu_{k2} < 0$, and the numbers $\mu_{k+1,r}/\mu_{kr}$ $(k = 0, \pm 1, \pm 2, \dots;\ r = 1, 2)$ are natural numbers distinct from one; b)* for some $K > 0$ the following holds:

$$\int_{|x|>y} dG(x) = O\left(\exp\{-Ky^2\}\right), \qquad y \to +\infty.$$

The purpose of the present paper is to indicate a sufficient condition for membership in I_0^α, essentially different from Yu. V. Linnik' s condition and, as it seems to us, of interest for the following reason. We shall call

* We note that in ([1]) (p. 242) condition b) is replaced by the stronger condition of boundedness of the set of

the Poisson spectrum of the infinitely divisible law F is the set of nonzero points of increase of the function $G(x)$ appearing in the representation of its characteristic function $\varphi(t, F)$ by the Lévy-Khintchine formula. One may pose the question: what closed set on the line can serve, after removal of the point 0, as the Poisson spectrum of a law of the class I_0^α? It follows from our result that it can be any perfect set, in particular the entire line. We note that for laws satisfying the sufficient condition indicated by Yu. V. Linnik, the Poisson spectrum is very sparse; it consists of isolated points that can accumulate only at 0 and ∞, and at a rate no less than that of a geometric progression with ratio 2*.

Moreover, it follows from our result that the class I_0^α is dense, in the sense of weak convergence, in the class of all infinitely divisible laws.

Let us formulate our main result.

Theorem 1. *Let F be an infinitely divisible law whose characteristic function has the form*

$$\varphi(t, F) = \exp\left\{i\beta t + \int_{-\infty}^{\infty} (e^{itx} - 1)\, dG(x)\right\} \tag{3}$$

where β is a real number, and $G(x)$ is a nondecreasing function satisfying the conditions:

*a) it is a step function, the points of discontinuity of which form a set with linearly independent points**;*

b) for some $K > 0$,

$$\int_{|x|>y} dG(x) = O(\exp\{-Ky^2\}), \quad y \to +\infty.$$

Then $F \in I_0^\alpha$.

This theorem is adjacent to Theorem 2 of our paper (4), where other conditions on the function $G(x)$ are indicated which ensure that the law with characteristic function of the form (3) belongs to the class I_0. The advantage of Theorem 1 is that it gives conditions for membership in I_0^α, and not in I_0, and, moreover, the Poisson spectrum of the law is not assumed to lie on the positive half-axis and to be bounded. However, Theorem 1 does not contain Theorem 2 of paper (4), since there the function $G(x)$ was not assumed to be a step function.

Let us note two corollaries of Theorem 1.

Corollary 1. *Let E be a closed set on the line, $E = P \cup R$, where P is a decomposition into disjoint perfect sets and at most countably many R sets. If R is either empty or is a set with linearly independent points, then there exists a law $F \in I_0^\alpha$ whose Poisson spectrum coincides with $E \setminus \{0\}$.*

Indeed, it is easy to see that in P one can indicate a countable dense set Q such that $Q \cup R$ is a set with linearly independent points. Let $Q \cup R = \{x_k\}$; let $G(x)$ be a step function whose set of discontinuity points coincides with $\{x_k\}$ and whose jump at the point x_k is equal to $2^{-k}e^{-x_k^2}$. Substituting this function into the right-hand side of (3), we obtain the desired law F.

Corollary 2. *The class I_0^α is dense in the class of all infinitely divisible laws in the sense that, for any infinitely divisible law F, one can indicate a sequence of laws $F_n \in I_0^\alpha$ such that F_n converges weakly to F as $n \to \infty$.*

* This is explained by the presence of a Gaussian component.

** A set A on the line is called, following D. A. Raikov, a set with linearly independent points if, for any finite collection of numbers $x_1, x_2, \dots, x_n \in A$, from the equality

$$h_1 x_1 + h_2 x_2 + \dots + h_{nx} n = 0,$$

where h_j are integers, it follows that

$$h_1 = h_2 = \dots = h_n = 0.$$

Indeed, let F be an infinitely divisible law, and let $G(x)$ be the function occurring in the Lévy-Khintchine representation (2). For each natural number n we construct a function $G_n(x)$ as follows. On each of the intervals $(m/n, (m+1)/n)$, $m = 0, \pm 1, \dots, \pm n^2$, choose points x_{mn} so that the set $A_n = \{x_{mn}\}_{m=-n^2}^{n^2}$ is a set of linearly independent points. Denote by $G_n(x)$ the step function whose discontinuities lie in A_n, such that $G_n(x_{mn}) = G(x_{mn})$, $m = 0, \pm 1, \dots, \pm n^2$. Let F_n be the law whose characteristic function is defined by the right-hand side of formula (2), where the role of $G(x)$ is played by $G_n(x)$. Then F_n satisfies the conditions of Theorem 1 and, as $n \to \infty$, converges weakly to F.

In ([6]) R. Cuppens reports that he has proved the following theorem. Let F be a law whose characteristic function has the form

$$\varphi(t, F) = \exp\left\{ i\beta t + \sum_{k=1}^{q} \sum_{s=1}^{r_k} c_{ks} \left(e^{i\lambda_{ks} t} - 1\right) + \sum_{j=1}^{\infty} d_j \left(e^{i\beta_j t} - 1\right) \right\},$$

where β is real, c_{ks} and d_j are nonnegative, and λ_{ks} and β_j are positive constants. Suppose that the following conditions are fulfilled:

1) the numbers $\lambda_{k,s+1}/\lambda_{ks}$ $(s = 1, \dots, r_k - 1;\ k = 1, \dots, q)$ and β_{j+1}/β_j $(j = 1, 2, \dots)$ are natural numbers different from one,
2) the numbers $\lambda_{11}, \lambda_{21}, \dots, \lambda_{q1}, \beta_1$ form a set of linearly independent points,
3) there exists a constant $K > 0$ such that

$$d_j = O(\exp\{-K\beta_j^2\}), \quad j \to \infty.$$

Then $F \in I_0$.

The method that we used to prove Theorem 1 makes it possible to obtain a more general result.

Theorem 2. *Let F be a law whose characteristic function has the form*

$$\varphi(t, F) = \exp\left\{ i\beta t + \int_{-\infty}^{\infty} \left(e^{itx} - 1\right) dG(x) \right\},$$

where β is a real constant and $G(x)$ is a nondecreasing function satisfying the conditions:

a) $G(x)$ is a step function and the set of its discontinuity points is contained in the set $\{\lambda_{ks}\}_{s=1}^{\infty}$, where the numbers $\lambda_{k,s+1}/\lambda_{ks}$ $(k, s = 1, 2, \dots)$ are natural numbers different from one, and the numbers $\lambda_{k1} > 0$ $(k = 1, 2, \dots)$ form a set of linearly independent points,

b) there exists a constant $K > 0$ such that

$$\int_{x>y} dG(x) = O(\exp\{-Ky^2\}), \quad y \to +\infty.$$

Then $F \in I_0$.

Let us note that if it were possible to dispense with the assumption $\lambda_{k1} > 0$ $(k = 1, 2, ...)$, and to replace the assertion $F \in I_0$ by the stronger $F \in I_0^a$, then we would obtain a theorem containing Theorem 1.

The method we use relies essentially on Bohr' s theory of almost periodic functions.

Physical-Technical Institute of Low Temperatures
Academy of Sciences of the Ukrainian SSR
Kharkov

Received
1 II 1970

REFERENCES

1. Yu. V. Linnik, *Decompositions of Probability Laws*, L., 1960.
2. I. V. Ostrovskii, Vestn. LGU, No. 19, 51 (1964).
3. I. V. Ostrovskii, Tr. Mat. inst. im. V. A. Steklova AN SSSR, 79, 198 (1965).
4. I. V. Ostrovskii, Vestn. Kharkov. gos. univ., 32, 51 (1966).
5. A. A. Gold' berg, I. V. Ostrovskii, Ukr. matem. zhurn., 19, 104 (1967).
6. R. Cuppens, C. R., 266, 726 (1968).
7. R. Cuppens, Proc. Am. Math. Soc., 21, No. 1, 145 (1969).
8. R. Cuppens, Pacif. J. Math., 29, 3, 521 (1969).
9. R. Shimizu, Ann. Inst. Stat., 16, 387 (1964).

Note: Figure translations are in progress. See original paper for figures.