AN \(n\)-DIMENSIONAL CUBE CANNOT BE DECOMPOSED INTO A COUNTABLE UNION OF PROPER CLOSED SUBSETS WHOSE PAIRWISE INTERSECTIONS ARE AT MOST \((n-2)\)-DIMENSIONAL

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Abstract

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N. HADZHIIVANOV (BULGARIA)

AN n-DIMENSIONAL CUBE CANNOT BE DECOMPOSED INTO A COUNTABLE UNION OF PROPER CLOSED SUBSETS WHOSE PAIRWISE INTERSECTIONS ARE AT MOST $(n-2)$-DIMENSIONAL

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The assertion announced in the title, for two summands, was proved by P. S. Urysohn in his famous paper "On Cantorian manifolds" (see (^{1}), p. 435).

Let an n-dimensional parallelepiped P be contained in a countable union of closed subsets Φ_k of the space E^n, $P \not\subset \Phi_k$, and $\dim(\Phi_i \cap \Phi_j) \leq n-2$ for any distinct i and j.

Lemma 1. *For every i, at least one of the summands Φ_k contains a continuum joining the opposite faces P_{-i} and P_{+i} of the parallelepiped P.*

By symmetry, it suffices to restrict ourselves to the case $i = 1$. Put

$$M = \bigcup_{i \neq j} (\Phi_i \cap \Phi_j).$$

The set M has dimension $\dim M \leq n-2$, while (P_{-i}, P_{+i}), $i = 2, 3, \dots, n$, are $(n-1)$ pairs of closed sets for which $P_{-i} \cap P_{+i} = \emptyset$. Then in P there exist closed partitions C_i between P_{-i} and P_{+i} such that the set

$$C = \bigcap_{i=2}^{n} C_i$$

does not intersect the set M.

To prove the lemma, it is enough to find at least one component of connectedness K of the compactum C joining P_{-1} with P_{+1}. Indeed, since $K \cap M = \emptyset$, the continuum K is decomposed into a union of pairwise disjoint closed sets $K \cap \Phi_k$. Hence, by the well-known theorem of Sierpiński (^{2}), $K = K \cap \Phi_k$ for some k, i.e. $K \subset \Phi_k$, as required by the lemma. Let us show that such a component K can be found. Indeed, if there were no such components, then, by a known assertion that every component of connectedness of a compactum is its quasicomponent, we could, for every component of connectedness of the compactum C, take a clopen neighborhood O in C not intersecting both faces P_{-1} and P_{+1} at the same time. These neighborhoods O cover the compactum C, and from them one can choose a finite subcover $O_1, O_2, \dots, O_p$. Denote by C_- the union of the face P_{-1} with all those neighborhoods O_i which do not intersect the face P_{+1}, and by C_+ the union of the face P_{+1} with the complement $C \setminus C_-$, which is clopen in C. It is clear that the closed sets C_- and C_+ do not intersect. Hence they can be separated in the parallelepiped P by a closed partition C_1. Thus the opposite faces of the parallelepiped P would be separated by partitions $C_1, C_2, \dots, C_n$ with empty intersection, which is impossible. Consequently, contrary to the supposition, there exists a component of connectedness K of the compactum C joining the face P_{-1} with the face P_{+1}, which was to be proved.

Lemma 2. *No summand Φ_k contains any parallelepiped N two faces of which are parallel to the faces P_{-i} and P_{+i}, while the remaining ones lie on the faces of the parallelepiped P.*

Of course, we may again assume that $i = 1$. Suppose that, for example, the summand Φ_1 contains entirely some parallelepiped N of the indicated···

of the prescribed form. Then the union of the closed differences $\Phi_k \setminus \operatorname{Int} N$, $k \neq 1$, together with the first summand Φ_1, covers the parallelepiped P; none of them contains P, and all their pairwise intersections are at most $(n-2)$-dimensional. Therefore one may assume that the given decomposition is such that a certain parallelepiped N of the indicated form does not meet any of the summands Φ_k, $k \neq 1$. Since the closed set Φ_1 does not contain P, there is in P a cube Q, with faces parallel to the faces of the parallelepiped P, which does not meet the set Φ_1. Extending its faces, parallel to the faces P_i, $i = 2, 3, \dots, n$, up to their intersection with the faces P_{-1} and P_{+1}, we obtain a parallelepiped R which contains Q. The parallelepiped R does not lie in any of the summands Φ_k, $k \neq 1$, since otherwise N would meet one of them. Nor does R lie in Φ_1, since the cube Q does not lie in Φ_1. Hence, by Lemma 1, there exists a continuum K lying in some $R \cap \Phi_k$ and joining the faces of the parallelepiped R that lie on the faces P_{-1} and P_{+1}. This continuum necessarily intersects the parallelepiped N, and consequently $\Phi_k \cap N \neq \emptyset$, which means that $k = 1$. Thus $K \subset \Phi_1$. But the continuum K also intersects the cube Q, so that $Q \cap \Phi_1 \neq \emptyset$, and this contradicts the choice of the cube Q. The lemma is proved.

Theorem. *An n-dimensional cube cannot be decomposed into a countable union of closed subsets distinct from the whole cube, whose pairwise intersections are no more than $(n-2)$-dimensional.*

Let the n-dimensional cube I^n be decomposed into a countable union of closed subsets Φ_k, $\Phi_k \neq I^n$, with $\dim(\Phi_i \cap \Phi_j) \leq n-2$ for any distinct i and j. Since I^n is a complete metric space, all summands Φ_k cannot be nowhere dense. Hence at least one of the summands contains entirely some cube Q with faces parallel to the faces of the cube I^n. Let this be the summand Φ_1. There exists a sequence of parallelepipeds P_i, $i = 1, 2, \dots, n+1$, with faces parallel to the faces of the cube I^n, the first of which coincides with I^n, and the last with Q, and, moreover, all faces of the parallelepiped P_{i+1}, except for one pair of opposite ones, lie on faces of the parallelepiped P_i. We obtain the parallelepiped P_n as follows. Extend all faces of the cube Q, except for one single pair of opposite faces, until they meet the faces of the cube I^n. The extended faces divide the cube I^n into 3^{n-1} parallelepipeds. P_n will be the one among them that contains Q. We construct the parallelepiped P_{n-1} in the same way, but this time starting from P_n and extending all its faces except for one single pair of opposite faces, different from those that lie on the faces of the cube I^n. It is clear that this process will stop at the $(n-1)$-st step, at which we obtain the parallelepiped P_2. All its faces, except for one single pair of opposite ones, will lie on faces of the cube I^n.

By Lemma 2, the parallelepiped P_2 is not contained entirely in any of the summands Φ_k. Then, by Lemma 2, P_3 is not contained in any of the summands Φ_k. Continuing this reasoning, at the n-th step we obtain that Q is not contained in any of the summands Φ_k, which contradicts the choice of the cube Q. The theorem is proved.

Corollary 1. *If a separable topological vector space L has a basis consisting of at least n elements, then L cannot be decomposed into a countable union of closed subsets, distinct from the whole L, whose pairwise intersections are no more than $(n-2)$-dimensional.*

This corollary is obtained quite easily from the theorem with the aid of the following lemma:

Lemma 3. *Let $\{X_\alpha\}$ be a family of such closed subsets of a space X that, for every α, X_α cannot be decomposed into a countable union of closed subsets distinct from X_α, whose pairwise intersections are no more than $(n-1)$-dimensional; then, if $\dim(X_{\alpha_0} \cap X_\alpha) \geq n$ for every $\alpha \neq \alpha_0$, then $X' = \bigcup_\alpha X_\alpha$ also cannot be decomposed into a countable union of the indicated form.*

Suppose there exists a decomposition $X' = \bigcup_{k=1}^{\infty} \Phi_k$, where the summands Φ_k are closed, $\Phi_k \neq X'$ for every k, and $\dim(\Phi_i \cap \Phi_j) \leq n-1$ for distinct i and j. Since X_α is not decomposed in this way, for every α there exists a Φ_{k_α} containing X_α. From the inclusion $X_{\alpha_0} \cap X_\alpha \subset \Phi_{k_{\alpha_0}} \cap \Phi_{k_\alpha}$ it follows that $\dim(\Phi_{k_{\alpha_0}} \cap \Phi_{k_\alpha}) \geq n$, and this entails $k_\alpha = k_{\alpha_0}$, i.e. $\Phi_{k_\alpha} = \Phi_{k_{\alpha_0}}$. Then $X_\alpha \subset \Phi_{k_{\alpha_0}}$ for every α and, consequently, $X' = \Phi_{k_{\alpha_0}}$, which is a contradiction. The lemma is proved.

Proof of Corollary 1. Let $l_1, l_2, \dots, l_n$ be distinct basis elements of the space L, and let a be an arbitrary element of the space L. Denote by L_a the convex hull spanned by the elements $a, l_1, l_2, \dots, l_n$. L_a is a closed subset of the space L, homeomorphic to an m-dimensional simplex T^m, where $m \geq n$. By the theorem, the set L_a is not decomposed into a countable union of closed subsets different from L_a, the pairwise intersections of which are at most $(n-2)$-dimensional.

Moreover, the intersection $L_{l_1} \cap L_a$ contains L_{l_1}, so that $\lim(L_{l_1} \cap L_a) \geq n-1$. Finally, $L = \bigcup_\alpha L_\alpha$. To complete the proof of Corollary 1, it remains only to apply Lemma 3.

Corollary 2. *If M is a connected topological space, every point x of which has a neighborhood homeomorphic to the $m(x)$-dimensional Euclidean space $E^{m(x)}$, where $m(x) \geq n$, then M is not decomposed into a countable union of closed subsets different from all of M, the pairwise intersections of which are at most $(n-2)$-dimensional.*

Let $M = \bigcup_{k=1}^{\infty} \Phi_k$, where $M \neq \Phi_k$ for every k, $\dim(\Phi_i \cap \Phi_j) \leq n-2$ whenever $i \neq j$, and the sets Φ_k are closed. Let $x \in M$ and let $\overline{O_x}$ be a closed neighborhood of the point x, homeomorphic to the cube $I^{m(x)}$, where $m(x) \geq n$. From the theorem it follows that $\overline{O_x}$ necessarily lies in one of the summands Φ_k, say in Φ_1. Hence $U = \operatorname{Int} \Phi_1 \neq \emptyset$. Since the space M is connected, $\operatorname{Fr} U \neq \emptyset$. Now let $y \in \operatorname{Fr} U$, and let $\overline{O_y}$ be a closed neighborhood of the point y, homeomorphic to the cube $I^{m(y)}$, where $m(y) \geq n$. There will be a summand Φ_k, $k \neq 1$, such that $\overline{O_y} \subset \Phi_k$. Put $F_1 = \Phi_1 \cap \overline{O_y}$ and $F_2 = \overline{O_y} \setminus U$. It is clear that $F_1 \neq \overline{O_y}$, $F_2 \neq \overline{O_y}$, $F_1 \cup F_2 = \overline{O_y}$, and $\dim(F_1 \cap F_2) \leq \dim(\Phi_1 \cap \Phi_k) \leq n-1$, which contradicts the theorem. Consequently, a decomposition of the space M of the indicated kind is impossible.

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.

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