ILLUMINATION FROM WITHIN FOR UNBOUNDED CONVEX BODIES

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ILLUMINATION FROM WITHIN FOR UNBOUNDED CONVEX BODIES

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Let K be a convex body of the n-dimensional Euclidean space E^n; let $\operatorname{bd} K$ be its boundary and $\operatorname{int} K$ its interior. A point $y \in \operatorname{bd} K$ will be called **illuminated from within** by a point $x \in \operatorname{bd} K$ if $x \neq y$ and the interval (x, y) is contained in $\operatorname{int} K$. A set $N \subset \operatorname{bd} K$ will be called **illuminated from within** by a set $M \subset \operatorname{bd} K$ if every point $y \in N$ is illuminated from within by some point $x \in M$. We shall assume throughout that $\operatorname{bd} K \neq \emptyset$.

Theorem 1. *The boundary* $\operatorname{bd} K$ *of a convex unbounded body* $K \subset E^n$ *is illuminated from within if and only if* K *is not a cone.*

Proof. If K is a cone with vertex a, then, evidently, the point a is not illuminated from within by any point $x \in \operatorname{bd} K$.

Conversely, if the boundary $\operatorname{bd} K$ of the body K is not illuminated from within, then there exists a point $z \in \operatorname{bd} K$ that is not illuminated from within by any point $x \in \operatorname{bd} K$. Let $y \in \operatorname{int} K$. If the ray zy did not belong entirely to the body K, then on this ray there would be found such a point $x \in \operatorname{bd} K$ that $y \in (x, z)$. But this would mean that the point z is illuminated from within by the point $x \in \operatorname{bd} K$, contradicting the assumption. Thus the ray zy is contained in the body K. Consequently, $\{z\} \cup \operatorname{int} K$ is a cone, and therefore, passing to the closure, we find that K is a cone (with vertex at the point z).

Theorem 1 resolves the question under what condition the entire boundary $\operatorname{bd} K$ of the body K is illuminated from within by some set $M \subset \operatorname{bd} K$. The least cardinality of an illuminating set M will be denoted by $p(K)$. It is known ([3],[5]) that for every bounded convex body $K \subset E^n$ the inequalities $2 \leq p(K) \leq n+1$ hold. Furthermore ([5]), for every natural p satisfying the inequalities $2 \leq p \leq n+1$, there exists such a bounded convex body $K \subset E^n$ that $p(K) = p$, and the equality $p(K) = n + 1$ holds only for the n-dimensional simplex. In this note the indicated results are extended to the case of unbounded convex bodies.

A point $x \in \operatorname{bd} K$ will be called **regular** if, first, through x there passes a **unique** supporting hyperplane Γ_x of the body K, and, second, the point x is an **interior point** of the convex set $\Gamma_x \cap \operatorname{bd} K$ (relative to the carrier plane of this set). If x is a regular point, then the closed convex set $F_x = \Gamma_x \cap \operatorname{bd} K$ will be called a **regular face**.

Lemma 1. *If x is a regular point and y is an interior point of the corresponding regular face F_x (relative to its carrier plane), then y is also a regular point and $F_y = F_x$.*

Lemma 2. *The set M of all regular points is dense in the boundary of the body K, i.e. $\overline{M} = \operatorname{bd} K$.*

For the proof of these lemmas see (^{2}), p. 110, and (^{1}), p. 24.

Lemma 3. *An unbounded convex body $K \subset E^n$ is a cone if and only if, for any n regular points $x_1, x_2, \dots, x_n$, the corresponding regular faces $F_{x_1}, F_{x_2}, \dots, F_{x_n}$ have a nonempty intersection.*

In the proof only the sufficiency of this condition is needed.

First of all it is established that if the intersection of any n regular faces is nonempty, then the intersection of all regular faces is nonempty (for this one considers all possible intersections of regular faces with one of the planes Γ_x, and applies Helly' s theorem (^{3}) to these intersections). Next it is shown (using Lemma 2) that if a is a common point of all regular faces, then $\operatorname{bd} K$ is a cone with vertex a. From this it follows without difficulty that K is a convex cone with vertex a.

Theorem 2. *For every unbounded convex body $K \subset E^n$ distinct from a cone, the inequalities $2 \le p(K) \le n$ hold.*

Proof. Since the body K is distinct from a cone, by Lemma 3 there exist n regular points $x_1, x_2, \dots, x_n \in \operatorname{bd} K$ such that the corresponding regular faces $F_{x_1}, F_{x_2}, \dots, F_{x_n}$ have empty intersection. If a point $x \in \operatorname{bd} K$ is not illuminated from within by some point x_i ($i = 1, 2, \dots, n$), then the whole segment $[x_i, x]$ lies on the boundary of the body K and, hence, in the hyperplane Γ_{x_i}. In particular, $x \in \Gamma_{x_i}$. Since, moreover, $x \in \operatorname{bd} K$, it follows that $x \in \Gamma_{x_i} \cap \operatorname{bd} K = F_{x_i}$. Thus, if a point $x \in \operatorname{bd} K$ is not illuminated from within by the point x_i, then $x \in F_{x_i}$, and consequently x belongs to the intersection of the faces $F_{x_1}, F_{x_2}, \dots, F_{x_n}$. But since the sets $F_{x_1}, F_{x_2}, \dots, F_{x_n}$ have empty intersection, it follows that every point $x \in \operatorname{bd} K$ is illuminated by at least one of the points $x_1, x_2, \dots, x_n$. Consequently, $p(K) \le n$.

The inequality $2 \le p(K)$ is obvious, since a single point $x \in \operatorname{bd} K$ cannot illuminate from within the whole boundary $\operatorname{bd} K$ of the body K (the point x itself will remain unilluminated from within). This completes the proof of Theorem 2.

Theorem 3. *For every integer p satisfying the inequalities $2 \le p \le n$, there exists an unbounded convex body $K \subset E^n$ such that $p(K) = p$.*

Proof of Theorem 3 is carried out by the same methods as for bounded convex bodies (see (^{5})), but cylinders are considered instead of balls. Namely, if L is a convex set with supporting hyperplane $\Gamma \subset E^n$ and if $a \notin \Gamma$, then, as is not hard to prove, for the cone aL with base L and vertex a the relation

$$p(aL) = p(L) + 1.$$

Therefore, if C is such a convex body of dimension $n-p+2$ for which $p(C) = 2$, then, forming $p-2$ times a cone over this body, we arrive at an n-dimensional body K satisfying the relation $p(K) = p$ (here p may take the values $2, 3, \dots, n$). In note (^{5}), an $(n-p+2)$-dimensional ball was used as C. If, however, one takes as C a cylinder, i.e., the r-neighborhood of a straight line in $(n-p+2)$-dimensional space, then this construction leads us to the required unbounded body. Thus Theorem 3 is proved.

Finally, let us note that an unbounded body $K \subset E^n$ for which $p(K)$ assumes its maximal possible value is no longer unique, as it was in the case of a bounded body. Indeed, if instead of C in the proof of Theorem 3 one considers some two-dimensional unbounded convex polygon having two unbounded parallel sides, then the body K to which we arrive, as is not hard to trace from the proof of Theorem 3, will satisfy the relation $p(K) = n$. But there are infinitely many such polygons C.

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