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Abstract

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THE METHOD OF LIMITING TRANSFORMATIONS IN THE PROBLEM OF THE EXISTENCE OF POISSON-STABLE SOLUTIONS OF DIFFERENTIAL EQUATIONS

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In the present note the question is solved of the existence of Poisson-stable solutions of differential equations of the form

$$dx/dt = f(t, x), \tag{1}$$

where f is a continuous mapping of the product $T \times X$ into the Banach space E, $T = (-\infty, +\infty)$, and $X \subset E$.

The concept of Poisson stability introduced by A. Poincaré ([1]) is defined as follows. A solution φ of equation (1), defined on T, is called **Poisson-stable in the positive (negative) direction** if, for any positive ε, s, and l, there exists $p \geq l$ $(p \leq -l)$ satisfying the condition

$$\sup_{<s} \|\varphi(t+p) - \varphi(t)\| < \varepsilon.$$

A solution φ is called **Poisson-stable** if it is Poisson-stable in the positive and negative directions. Geometrically, Poisson stability in the positive (negative) direction of the solution φ means that on its graph, to the right (left) of any given arc of it, there are arcs repeated that differ from the given one by an arbitrarily small amount. This characteristic geometric feature of Poisson stability is called recurrence.

It follows directly from the definitions given that Poisson stability is the broadest generalization of the concept of periodicity, and solutions of differential equations possessing this kind of stability describe certain generalized oscillatory regimes. As A. Poincaré showed ([1]), such regimes, distinct from periodic ones, are observed not only in problems of celestial mechanics but also in a number of physical phenomena, such as, for example, the stationary motion of an incompressible fluid.

The problem mentioned below is solved by means of a method previously applied in ([2,3]) in the search for recurrent solutions. At the same time, no conditions are imposed on the equations under consideration that would ensure the existence of solutions for all initial values. It is also not assumed that the solutions under study are uniquely determined by their initial values.

1°. Let us introduce some auxiliary notions. Let X be a metric space with distance ρ. By the symbol $C(I;X)$, where I is an interval of the straight line T, we shall denote the set of all continuous functions defined on I and with values in X. Put

$$C^*(I;X) = \bigcup_{I \subset T} C(I;X),$$

where the union is taken over all, including unbounded, intervals I of the straight line T.

Let, for each natural number n, a function φ_n be given, defined on $I_n = (a_n, b_n)$. We shall say that the sequence $\{\varphi_n\}$ converges if there exists a function $\varphi \in C^*(I;X)$, defined on $I = (a,b)$, such that the following conditions are fulfilled: 1) $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$; 2) for each ⋯

whatever the number $\varepsilon > 0$ and the segment $S \subset I$, there exists an m such that for all $n \geqslant m$ the inequality

$$\sup_{t\in S} \rho(\varphi_n(t), \varphi(t)) < \varepsilon$$

holds. When these conditions are satisfied, the function φ is called the limit of the sequence $\{\varphi_n\}$, and this fact is written in the form of the equality $\lim_{n\to\infty} \varphi_n = \varphi$.

The set $C^*(I;X)$, for which the indicated class of convergent sequences has been singled out, is an $\mathcal{L}^*$-space ([4]). We note that every subset of the space $C^*(I;X)$ may be regarded as a certain independent $\mathcal{L}^*$-space.

Let $\varphi \in C^*(I;X)$, let I be the interval of definition of the function φ, and let $p \in T$. By the symbol $\varphi^{(p)}$ we shall denote the shift of the function φ by p, i.e.

$$\varphi^{(p)}(t) = \varphi(t+p)$$

for all t from the set $\{s-p;\ s \in I\}$. We shall say that the numerical sequence $\{p_n\}$ directs φ to $\psi \in C^*(I;X)$ if

$$\lim_{n\to\infty} \varphi^{(p_n)} = \psi.$$

For each function $\varphi \in C^*(I;X)$ and each numerical sequence $\{p_n\}$, define the set $W(\varphi, \{p_n\})$ as follows: a function $\psi \in C^*(I;X)$ belongs to $W(\varphi, \{p_n\})$ if and only if from $\{p_n\}$ one can extract a subsequence directing φ to ψ.

A numerical sequence $\{p_n\}$ will be called an ω-sequence (an α-sequence) if

$$0 < p_1 < p_2 < \cdots \quad (0 > p_1 > p_2 > \cdots)$$

and

$$\lim_{n\to\infty} p_n = +\infty \quad \left(\lim_{n\to\infty} p_n = -\infty\right).$$

It follows from the definitions given that, for any function $\varphi \in C^*(I;X)$ and any ω- or α-sequence $\{p_n\}$, the set $W(\varphi, \{p_n\})$ is contained in $C(T;X)$.

2°. We indicate some features of the limiting behavior of solutions of differential equations of the form

$$dx/dt = F(t)x, \tag{2}$$

where F is a continuous mapping of some interval of the line T into the space R of all continuous mappings of some compact $Q \subset E$ into E. Convergence in R is uniform convergence of functions on Q.

Lemma 1. The set of all solutions of equation (2) is closed and compact in the space $C^*(I;Q)$.

Lemma 2. Suppose that the right-hand side F of equation (2) is defined on an interval unbounded in the positive (negative) direction, and let $\{p_n\}$ be an ω-sequence (an α-sequence) directing F to a certain function $G \in C(T;R)$. Then, for any solution φ of equation (2), defined on an interval unbounded in the positive (negative) direction, the set $W(\varphi;\{p_n\})$ is nonempty, closed, and compact in the space $C(T;Q)$ and consists of solutions of the differential equation

$$dx/dt = G(t)x, \tag{3}$$

defined on the whole line T.

In the proof of Lemmas 1 and 2, Ascoli' s theorem ([5]) is used, containing criteria of compactness for spaces of continuous functions.

Suppose the assumptions of Lemma 2 are satisfied. Then, according to this lemma,

$$W = W(\varphi, \{p_n\})$$

is a certain limiting transformation of solutions of equation (2), defined on intervals unbounded in the positive (negative) direction, into solutions of equation (3), defined on the whole line T. This fact plays an essential role in what follows.

3°. We give some criteria for the existence of solutions of differential equations that are stable in the sense of Poisson.

We shall say that the right-hand side f of equation (1) is **Poisson stable in the positive (negative) direction**

on the set $Q \subset X$, if for any positive ε, s and l there exists a $p \geq l$ ($p \leq -l$) such that

$$\sup_{|t|\leq s,\ x\in Q} \|f(t+p,x) - f(t,x)\| < \varepsilon.$$

We shall call the function f **Poisson stable on** Q if it is Poisson stable in the positive and negative directions on Q.

A solution φ of equation (1), defined on I, is called **compact** if there exists a compact set $Q \subset X$ such that the set $\varphi(I) = \{\varphi(t);\, t \in I\}$ is contained in Q.

Theorem 1. *Suppose that the right-hand side f of equation (1) is Poisson stable on every compact set $Q \subset X$. In order that every differential equation of the form*

$$dx/dt = f(t+p, x), \tag{4}$$

where p is an arbitrary real number, admit Poisson-stable solutions, it is sufficient that there exist at least one compact solution of equation (1), defined on an unbounded interval.

For the proof of the theorem, by means of limiting transformations one constructs a transfinite sequence of solutions of equation (4), $\varphi_1, \varphi_2, \ldots, \varphi_\omega, \varphi_{\omega+1}, \ldots$, defined on the whole line T and satisfying the condition

$$\overline{\Sigma}_{\varphi_1} \supset \overline{\Sigma}_{\varphi_2} \supset \cdots \supset \overline{\Sigma}_{\varphi_\omega} \supset \overline{\Sigma}_{\varphi_{\omega+1}} \supset \cdots,$$

where $\overline{\Sigma}_{\varphi_\gamma}$, for each ordinal γ, is the closure of the set

$$\Sigma_{\varphi_\gamma} = \{\varphi_\gamma^{(t)};\ t \in T\}$$

in the space $C(T; X)$. Then the Baire-Hausdorff theorem is applied.

Remark 1. According to Theorem 1, under the assumption stated in it concerning f, every criterion for the existence of at least one compact solution of equation (1), defined on an unbounded interval, is at the same time a criterion for the existence of Poisson-stable solutions of every equation of the form (4). Thus, for the question of existence of the solutions of interest to us, Theorem 1 makes it possible to use numerous results on the nonlocal extendability and compactness of solutions of the equations under consideration. We also note that there are criteria, analogous to Theorem 1, for the existence of solutions Poisson stable in one direction.

Let us now consider the question of the existence of Poisson-stable solutions of equation (1) that are of the same type, in the sense of the frequency of recurrence, as the right-hand side f. We shall call such solutions resonant. Let us make this concept precise. A solution φ of equation (1), defined on T, will be called **resonant** if for any $\varepsilon > 0$ and $s > 0$ there exist $\delta > 0$ and $r > 0$ such that every real p for which the inequality

$$\sup_{|t|\le r,\ x=\varphi(T)} \|f(t+p, x) - f(t, x)\| < \delta$$

holds also satisfies the condition

$$\sup_{|t|\le s} \|\varphi(t+p) - \varphi(t)\| < \varepsilon.$$

It is not difficult to see that if the right-hand side f of equation (1) is periodic in t (uniformly with respect to $x \in X$), and p is its period, then a solution φ

of equation (1), defined on T, is resonant if and only if φ is a periodic solution with period p. In this case the problem of the existence of resonant solutions reduces to the problem of the existence of periodic solutions with period p. An analogous conclusion is valid for the case when the right-hand side f of equation (1) is almost recurrent (6), N-almost-periodic (7), and for a number of other types of Poisson stability of the function f.

A solution ψ of equation (1) will be called an **ω-limit** (**α-limit**) **image** of a solution φ of the same equation if there exists an ω-sequence (an α-sequence) directing φ to ψ.

Theorem 2. *Suppose that the right-hand side f of equation (1) is Poisson stable in the positive (negative) direction on every compact set $Q \subset X$. In order that a compact solution φ of equation (1),*

defined on T, to be resonant, it is sufficient that equation (1) admit no solutions distinct from φ which are ω-limit (α-limit) images of the solution φ.

Theorem 2 is proved with the aid of limiting transformations of the set of compact solutions of equation (1), defined on T, into itself.

Let us note that the sufficient condition of Theorem 2 is fulfilled if, for example, for every solution $\psi \neq \varphi$ of equation (1), defined on T, the set $\psi(\overline{T})$ is not contained in $\varphi(\overline{T})$. The latter holds if, for example, φ is the unique compact solution of equation (1), defined on T, or if φ is the unique minimal solution of the same equation (here minimality is understood in the same sense as in (7)).

From Theorem 2 there follows a series of new criteria for the existence of various types of Poisson-stable solutions of differential equations. For example, a consequence of Theorem 2 is the following proposition, generalizing the theorems of B. M. Levitan ([7]) on the existence of N-almost-periodic solutions of a system of n linear equations with almost-periodic coefficients.

Theorem 3. *Suppose that the right-hand side f of equation (1) is N-almost-periodic in t (uniformly with respect to $x \in X$). In order that a compact solution φ of equation (1), defined on T, be N-almost-periodic, it is sufficient that, for every solution ψ of equation (1), defined on T and distinct from φ, the set $\psi(\overline{T})$ not be contained in $\varphi(\overline{T})$.*

Another example of a proposition following from Theorem 2 may be

Theorem 4. *Let A be a real matrix of order n whose spectrum does not intersect the imaginary axis. Whatever the continuous bounded mapping g of the line T into the n-dimensional space E_n, the differential equation*

$$dx/dt = Ax + g(t) \tag{5}$$

admits a unique bounded resonant solution.

Theorem 4 may be regarded as an extended Bohr–Neugebauer criterion ([8]), extending not only to almost-periodicity, but also to the other types of stability in the sense of Poisson.

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CITED LITERATURE

[1] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste* 3, Paris, 1899.
[2] B. A. Shcherbakov, DAN, 167, No. 5 (1966).
[3] B. A. Shcherbakov, *Differential Equations*, 3, No. 9 (1967).
[4] K. Kuratowski, *Topology*, 1, Moscow, 1966.
[5] N. Bourbaki, *Topologie générale*, Livre 3, ch. 10, *Espaces fonctionnels*, Paris, 1953.
[6] M. V. Bebutov, Byull. MGU, 2, no. 5 (1941).
[7] B. M. Levitan, *Almost-periodic functions*, Moscow, 1953.
[8] H. Bohr, O. Neugebauer, Nachr. Göttingen, Math.-phys. Klasse (1926).

Note: Figure translations are in progress. See original paper for figures.