THE RELATION BETWEEN TOPOLOGICAL ENTROPY AND METRIC ENTROPY

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THE RELATION BETWEEN TOPOLOGICAL ENTROPY AND METRIC ENTROPY

(Presented by Academician A. N. Kolmogorov, 29 V 1969)

This article studies the connection between the notions of topological entropy, ε-entropy, and metric entropy. Some examples are considered.

I. The connection between topological entropy and ε-entropy. Let $M = (E, \rho)$ be a compact metric space (E is the set of points, $\rho(x, y)$ is the distance), and let $T : M \to M$ be an arbitrary homeomorphism. Denote by $h(T \mid M)$ the topological entropy of the transformation T on the space M ([4]). Introduce on E new metrics

$$\rho_n(x, y) = \max_{0 \le i < n} \rho(T^i x, T^i y).$$

Let $H_n(\varepsilon)$ be the ε-entropy of the space $M_n = (E, \rho_n)$ in the sense of ([1]). A. N. Kolmogorov (unpublished) suggested the idea of relating the topological entropy of a homeomorphism with the asymptotics of the function $H_n(\varepsilon)$.

Theorem 1. *The equality holds*

$$h(T \mid M) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{H_n(\varepsilon)}{n}.$$

The proof of this theorem is not difficult, and therefore we do not give it. Theorem 1 often proves useful in estimating from below the topological entropy of a particular dynamical system (see below).

II. The connection between topological entropy and metric entropy. All measures considered below are assumed to be normalized and defined on the σ-algebra of all Borel subsets of some compact metric space.

We shall denote by $h_\mu(T \mid M)$ the metric (with respect to the measure μ) entropy of the transformation T on M, and by $h_\mu(T \mid M; \xi)$ the metric (with respect to the measure μ) entropy per unit time of a partition ξ of the space M ([2]).

In ([4]) the hypothesis was put forward that

$$h(T \mid M) = \sup_{\mu} h_{\mu}(T \mid M),$$

where the supremum is taken over all Borel measures μ invariant with respect to T. Below the validity of this hypothesis is proved for homeomorphisms of an arbitrary compact space of finite dimension.

Theorem 2. *Let M be an arbitrary compact metric space of finite dimension m, and let $T : M \to M$ be its homeomorphism. Then*

$$h(T \mid M) = \sup_{\mu} h_{\mu}(T \mid M),$$

where the supremum is taken over all Borel measures μ invariant with respect to T.

Proof. First we shall prove that $h(T \mid M) \leq \sup h_{\mu}(T \mid M)$. Suppose that $h(T \mid M) < \infty$. (The case $h(T \mid M) = \infty$ is treated similarly.) It is enough to prove that for every positive integer r there exists a Borel measure μ_r, invariant with respect to T^r, such that

$$h_{\mu_r}(T^r \mid M) \geq h(T^r \mid M) - b_r = rh(T \mid M) - b_r,$$

where $|b_r| < b$. Indeed, the measure

$$\mu_r' = \frac{1}{r} \sum_{i=0}^{r-1} T^i \mu_r$$

is invariant with respect to T, and

$$h_{\mu_r'}(T \mid M) = \frac{1}{r} h_{\mu_r'}(T^r \mid M) \geq \frac{1}{r} h_{\mu_r}(T^r \mid M) \geq h(T \mid M) - \frac{b_r}{r}.$$

The desired inequality follows from the fact that $b_r/r \to 0$ as $r \to \infty$.

For any finite covering $\{V_i\}$ of the space M by closed sets $V_1, \dots, V_N$, consider the set A of all sequences $\{i_k\}$ $(-\infty < k < \infty)$, composed of the symbols $1, \dots, N$, such that

$$\bigcap_{-\infty}^{\infty} T^{kr} V_{i_k}$$

is nonempty.

Lemma 1. The set A is closed in the space Ω^N of all sequences composed of the symbols $1, \dots, N$, and is invariant with respect to the shift S by one symbol to the right.

Lemma 2. There exists $\delta(T, r)$ such that, if $\operatorname{diam}\{V_i\} < \delta(T, r)$,

$$h(S \mid A) > h(T^r \mid M) - 1.$$

Lemma 3. On A there exists a Borel measure μ, invariant with respect to S, such that

$$h_\mu(S \mid A) = h(S \mid A).$$

(See also ([8]).)

The central point of the proof is the following assertion.

Lemma 4. There exist a covering $\{V_i\}$ of the space M by closed sets $V_1, \dots, V_N$, a compact Ω, and a homeomorphism $R : \Omega \to \Omega$ such that:

1) each element of the covering $\{V_i\}$ intersects no more than 3^{2m+1} elements of the same covering;
2) the space of sequences A, constructed by the method described above from the covering $\{V_i\}$, satisfies the conditions of Lemmas 1-3;
3) there exists a continuous mapping $p : \Omega \to A$ such that $p(\Omega) = A$ and $pR = Sp$;
4) there exists a continuous mapping $p_0 : \Omega \to M$ such that $p_0(\Omega) = M$ and $p_0 R = T^r p_0$;
5) for any sequence $\omega = \{i_k\} \in A$,

$$p_0 p^{-1}(\omega) \subset \bigcap_{-\infty}^{\infty} T^{kr} V_{i_k}.$$

Lemma 5. On Ω there exists a Borel measure ν, invariant with respect to R, such that for every Borel set $B \subset A$,

$$\mu(B) = \nu(p^{-1}(B)).$$

Define a Borel measure μ_r on M by setting $\mu_r(B) = \nu(p_0^{-1}(B))$ for every Borel set $B \subset M$. The measure μ_r has the required properties. Indeed, from Lemma 4 it follows that the measure μ_r is invariant with respect to T^r. Consider the partition α of the space A into the sets $C_1, \dots, C_N$, where $C_i = \{\omega = \{j_k\} : j_0 = i\}$. Clearly,

$$h(S \mid A) = h_\mu(S \mid A; \alpha) = h_\nu(R \mid \Omega; \xi),$$

where $\xi = p^{-1}(\alpha)$.

Let β be the partition of the space M into the sets $\widetilde{C}_1, \dots, \widetilde{C}_N$, where

$$\widetilde{C}_1 = V_1, \qquad \widetilde{C}_i = V_i - \bigcup_{k=1}^{i-1} \widetilde{C}_k \quad \text{for } 1 < i \le N.$$

Put $\eta = p_0^{-1}(\beta)$. Clearly,

$$h_{\mu_r}(T^r \mid M; \beta) = h_\nu(R \mid \Omega; \eta).$$

Further:

$$h_\nu(R \mid \Omega; \eta) = \lim_{n\to\infty} \frac{1}{n} H(\eta^n) = \lim_{n\to\infty} \left[\frac{1}{n} H(\xi^n \eta^n) - \frac{1}{n} H(\xi^n/\eta^n)\right] \ge$$

$$\ge h(S \mid A) - \lim_{n\to\infty} \frac{1}{n} H(\xi^n/\eta^n) \ge h(S \mid A) - (2m+1)\log 3.$$

Consequently,

$$h_{\mu_r}(T^r \mid M) > h(T^r \mid M) - b_r, \qquad \text{where} \quad |b_r| < (2m+1)\log 3 + 1.$$

Lemma 6. *For any metric compactum M of dimension m ($m < \infty$), any homeomorphism $T : M \to M$, and any Borel measure μ invariant with respect to T, the inequality $h(T \mid M) \ge h_\mu(T \mid M)$ holds.*

The proof of Lemma 6 follows from the fact that in any open cover of the compactum M one can inscribe a cover each element of which intersects no more than 3^{2m+1} elements of the given cover.

Theorem 2 follows from Lemmas 1-6.

G. A. Margulis (unpublished) independently proved Lemma 6 for compacta of a more general nature.

Not for every homeomorphism T of a compactum M does there exist an invariant measure μ such that $h(T \mid M) = h_\mu(T \mid M)$. In [9] an example is constructed of a homeomorphism of a zero-dimensional compactum that does not have this property.

Below it is assumed that M is a metric space of finite dimension.

Corollary 1. *The entropy $h(T \mid M)$ is equal to the topological entropy of the restriction of the transformation T to the set of nonwandering points.*

This assertion in a more general situation was proved by Bowen [5].

Corollary 2. *For a flow $\{S_t\}$ on M, for any t*

$$h(S_t \mid M) = |t| h(S_1 \mid M).$$

Corollary 2 was previously proved by A. M. Stepin (unpublished).

Corollary 3. *Let $M = X \times Y$, and let $T : M \to M$ be a homeomorphism identical on the base X, i.e. $T(x, y) = (x, T_x y)$. Then*

$$h(T \mid M) = \sup_x h(T_x \mid Y).$$

This assertion is a simple consequence of the fact that the equality $h(T \mid M) = \sup h_\mu(T \mid M)$ also holds in the case when the supremum is taken only over all invariant ergodic Borel measures μ.

Example A. The topological entropy of the geodesic flow on a compact surface of constant negative curvature coincides with its metric entropy with respect to the usual Riemannian measure.

The topological entropy of the horocycle flow on such a surface is equal to zero.

Example B. Let M be an arbitrary compact orientable surface of genus $p > 1$, let $\widetilde{M}$ be the space of line elements to M, and let S_t be the geodesic flow on $\widetilde{M}$.

Theorem 3. *The topological entropy $h(S_t \mid \widetilde{M})$ is positive, and there exists a measure μ, invariant with respect to S_t, for which $h_\mu(S_t \mid \widetilde{M}) > 0$.*

The proof is based on the study of geodesics of class A and on results of M. Morse concerning such geodesics [6, 7].

Example C. Motion of a particle on a torus in the field of a repelling potential. Let the motion of a point be defined by Hamilton' s equations with Hamiltonian function $H = \dot{x}^2/2 + \dot{y}^2/2 + U(r)$, where $U(r)$ is a smooth function for $r > 0$, with $U(r) \equiv 0$ for $r \geq r_0$, $\partial U/\partial r \leq 0$, and $U(r) \to \infty$ as $r \to 0$ (here $r^2 = x^2 + y^2$, and the coordinates x, y are considered modulo 1).

Theorem 4. For every fixed $\bar{H} \leq H_0$ and $r_0 \leq r(H_0)$, the flow defined by the motion of a point on the surface of constant energy $H = \bar{H}$ has positive topological entropy.

In the work of Ya. G. Sinai (3), conditions were found under which, for systems of this type, the Lebesgue measure has positive entropy.

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Note: Figure translations are in progress. See original paper for figures.