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## Abstract

## Full Text

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*MATHEMATICS*

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# ON MULTIPLE REGULAR BASES IN KÖTHE SPACE

*(Presented by Academician L. V. Kantorovich on 26 IX 1969)*

1. Let $E$ be a Köthe space*; let $(x_n)$ be an arbitrary absolute basis of it; let $(y_n)$ be one of those absolute bases in $E$ which can be obtained as a result of the following three operations: a) the mapping $(x_n) \to (Tx_n)$, where $T$ is an isomorphism of the space $E$ onto itself; b) multiplication of the elements $Tx_n$ by numbers $\lambda_n$, $\lambda_n \neq 0$ $(n = 1, \ldots)$; c) permutation of the terms of the sequence $(\lambda_n Tx_n)$. The bases $(x_n)$ and $(y_n)$ are called, respectively, equivalent, pre-equivalent, and quasi-equivalent. Up to the present time the question has not been completely solved (see, for example, ($^1$)): in what case are all absolute bases of a space in one of the listed equivalence relations?

It is not difficult to see that any two bases in $E$ are equivalent if and only if the space is finite-dimensional (here it is essential that the topology in $E$ can be given by a countable set of norms ($^3$)). Analogously, all absolute bases in $E$ are pre-equivalent if and only if $E$ is normable (i.e., degenerates into the Banach space $l_1$)**. It is not known, however, whether all absolute bases of a nondegenerate space $E$ are quasi-equivalent. A positive answer has been obtained only for some special classes of nuclear and Montel Köthe spaces possessing a regular*** basis ($^{3-9}$).

In the present article a more general case is considered. Let $(x_n)$ be an arbitrary sequence in $E$; let $\mu$ be the minimal number of its regular subsequences such that each element $x_n$ is contained in one and only one of them. Put $m = \inf \mu$, where the infimum is taken over the set of all possible permutations of the elements $x_n$ $(n = 1, 2, \ldots)$. A sequence for which $m \neq 1$ cannot be made regular by a permutation of its elements. It is natural to call it an $m$-fold regular sequence if $m$ is finite and $\mu = m$. We shall extend this definition also to the case when $m = 1$ or $m = \infty$. As will be shown, the multiplicity $m$ of an arbitrary basis in a nuclear space $E$ depends only on the space. In other words, each nuclear Köthe space belongs to one and only one of the classes: $\mathcal{N}_s = \{E : m = s\}$,

$1 \leq s \leq \infty$. All the classes $\mathcal{N}_s$ are nonempty. Any of them, with the exception, perhaps, of $\mathcal{N}_\infty$, contains spaces all of whose bases are quasi-equivalent.

2. Let $(x_n)$ be a sequence in $E$, the elements of which are not equal to zero; let $(t_n) = t$ be an arbitrary numerical sequence and

$$|t|_p = \sum_n |t_n| \, \|x_n\|_p \qquad (p = 1, 2, \ldots).$$

We agree to denote by $[x_n]$ the Köthe space

$$\{t : \ |t|_p < \infty, \ p = 1, \ldots\}$$

(with the topology given by the system—

by the system of norms $|\cdot|_p$. If $E$ is a Montel space with an absolute regular basis $(x_n)$, then denote

$$K(E) = \{[\lambda_n x_{k_n}] : \ k_n \to \infty, \ \lambda_n > 0, \ n = 1, \ldots\}.$$

Suppose that all regular bases in $E$ are pre-equivalent (in this case $K(E)$ does not depend on the choice of basis) and, moreover, that every space contained in $K(E)$ has the analogous property. The class $\mathcal{E}$ of all spaces $E$ under consideration is nonempty (it contains, in particular, spaces of type $(d_i)$, $i = 1, 2$ ($^7$)). Moreover, $E \in \mathcal{E}$ if and only if $K(E) \subset \mathcal{E}$.

For arbitrary spaces $E_i \in \mathcal{E}$ $(i = 1, 2)$, put

$$K(E_1) \oplus K(E_2) = \{G_1 \oplus G_2 : \ G_i \in K(E_i), \ i = 1, 2\}.$$

Let $R$ be a subset of the class $\mathcal{E}$, closed with respect to the operation $\oplus$. We say that the function $K(E)$ is additive on $R$, if $K(E_1 \oplus E_2) = K(E_1) \oplus K(E_2)$ for any $E_i \in R$ $(i = 1, 2)$.

**Definition 1.** A maximal subset $R \subset \mathcal{E}$, closed with respect to the operation $\oplus$, will be called a **Riss class** if the function $K(E)$ is additive on $R$.

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* A Köthe space is a complete countably normed space with an absolute Schauder basis.

** A more particular assertion is proved in ($^{10}$).

*** A sequence $(x_n)$, $x_n \in E$ $(1 \leq n < \nu \leq \infty)$, is called regular if there exists a defining system of norms in $E$ such that, for any $p, q = 1, 2, \ldots$, the sequence of numbers $\|x_n\|_p / \|x_n\|_q$ $(1 \leq n < \nu \leq \infty)$ is monotone ($^7$).

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For example, the set $R_0$ (respectively, $R_\infty$) of all Köthe spaces that are finite (infinite) centers of Riss scales ($^5$) is a Riss class. Let, in general, $E$ be an arbitrary space of type $(d_1)$ or $(d_2)$. The following holds.

**Theorem 1.** *The set $K(E)$ is a Riss class if and only if $E$ is isomorphic to each of its subspaces of finite codimension.*

In what follows we call a Riss class **complete** if there exists a space $E \in R$ such that $K(E) = R$.

Let us note some corollaries. Let $f(u)$ be a nondecreasing odd function of a real argument, logarithmically convex for $u \geq 0$. In ($^7$) a broad class of Montel Köthe spaces $L_f(b, r) = [(\delta_{nj})_{j=1}^{\infty}]$ is considered, for which

$$|(\delta_{nj})_{j=1}^{\infty}|_p = \exp f(r_p b_n) \quad (r_p \uparrow r),$$

where $-\infty < r \leq \infty$, $b = (b_n)$, and $b_n \uparrow \infty$. By definition,

$$(f)_\sigma = \{L_f(b, r) : \ b_n \uparrow \infty, \ r = \sigma\} \quad (\sigma = -1, 0, 1, \infty).$$

There, countable families of pairwise disjoint classes $(f)_\sigma$ are also singled out.

**Corollary 1.** *The set $(f)_\sigma$ is a complete Riss class.*

**Corollary 2.** *The cardinality of a maximal set of pairwise disjoint complete Riss classes is at least $c$.*

**Corollary 3.** *The set $\mathfrak{R}$ of all Riss classes is uncountable.*

We formulate necessary and sufficient conditions for the classes $R_i \in \mathfrak{R}$ $(i = 1, 2)$ not to intersect. Let $T : E_1 \to E_2$ be a continuous linear operator that carries an absolute basis $(x_n)$ of the space $E_1 \in R_1$ into an absolute basis of the space $E_2 \in R_2$.

**Theorem 2.** *The following assertions are equivalent:*

1°. $R_1 \cap R_2 = \emptyset$.

2°. *Whatever the spaces $E_i \in R_i$ $(i = 1, 2)$, the absolute basis $(x_n)$ in $E_1$, and the operator $T$, there exists a subsequence $(x_{j_n}, n = 1, 2, \ldots)$ such that the restriction of $T$ to the corresponding subspace*

$$\operatorname{span}(x_{j_n}) \subset E_1$$

*is compact.**

If the classes $R_1$ and $R_2$ are such that all continuous linear mappings of each space $E_1 \in R_1$ into any space $E_2 \in R_2$ are compact, then we shall write $R_1 > R_2$.

**Definition 2.** Classes $R_i \in \mathfrak{R}$ $(i = 1, 2)$ will be called **essentially distinct** if $R_1 > R_2$ or $R_2 > R_1$.

* For the definition of a compact operator, see, for example, ([11]).

For example, the classes $R_0$ and $R_\infty$ are essentially different, and moreover $R_0 > R_\infty$ ([2]). As V. P. Zaharyuta showed (ibid.), there exist countable families of Riss classes $(f)_\sigma$, linearly ordered by the relation $(\geq)$. It is not known whether any nonintersecting classes $R_i \in \mathfrak{R}$ $(i = 1, 2)$ are essentially different.

**Remark.** Nonintersecting subsets $K(E_i) \subset R_i$ $(i = 1, 2)$ cannot be essentially different in the sense of Definition 2.

Let, further, $E = \bigoplus E_i$, where $E_i \in R_i \in \mathfrak{R}$ $(i = 1, \dots, k)$, and $R_i = K(E_i)$ for $i \leq s \leq k$, i.e. all the Riss classes $R_1, \dots, R_s$ are complete, and the corresponding spaces $E_i$ satisfy the condition of Theorem 1. Important for what follows is

**Theorem 3.** *If $1 \leq s \leq k$ and all $R_i$ are distinct (respectively, $1 \leq s < k$ and $R_i \cap R_j = \emptyset$ for $i \leq s < j$), then in the space $E$ there exists an $m$-fold regular basis, for which $m \geq s$ (respectively $m \geq s + 1$). Moreover, equality is attained in every case.*

3. Let $E \subset \mathcal{N}$, where $\mathcal{N}$ is the class of all nuclear Köthe spaces. As is known ([12]), all bases in $E$ are absolute.

**Lemma** (cf. ([7],[8])). *Whatever the bases $(x_n)$ and $(y_n)$ in $E$, there exist two sequences $(k_n)$, $k \to \infty$, and $(\lambda_n)$, $\lambda_n > 0$ $(n = 1, 2, \dots)$, such that $[y_n] = [\lambda_n x_{k_n}]$.*

We give a number of corollaries, some of which are of independent interest.

**Corollary 1.** *In a nuclear space with a regular basis, the multiplicity $m$ of an arbitrary basis is equal to one.*

**Corollary 2.** *In a nuclear space $E \in \mathcal{E}$, all bases are quasiequivalent.*

Extend to the whole class $\mathcal{N}$ the function $K(E)$ defined on the set $\mathcal{N} \cap \mathcal{E}$ (here, as $(x_n)$, an arbitrary basis of the space is taken).

**Corollary 3.** *The function $K(E)$ does not depend on the choice of a basis in $E$ and is a topological invariant on the class $\mathcal{N}$.*

**Corollary 4.** *The multiplicity $m$ of an arbitrary basis in a nuclear space $E$ is a constant depending only on $E$.*

**Corollary 5.** *Every space $E \in \mathcal{N}$ belongs to one and only one of the classes $\mathcal{N}_s$ $(1 \leq s \leq \infty)$.*

Hence, and also from Theorems 1 (Corollary 2) and 3, it follows that

**Corollary 6.** *None of the classes $\mathcal{N}_s$ $(1 \leq s \leq \infty)$ is empty.*

Let, further, $E^j$ denote the subspace of the space $E$ of codimension $j$ for $j \geq 0$, and the direct sum of the form $E \oplus L$, where $\dim L = -j$, for $j < 0$. From Corollary 3 there follows a theorem of conditional type.

**Theorem 4.** *Let* $E = \bigoplus E_i$, *where* $E_i \in R_i$ $(i = 1, \dots, s)$ *are nuclear spaces, and* $R_i \cap R_k = \emptyset$ *if* $i \neq k$. *The following assertions are equivalent:*

1°. *In the space* $E$ *all bases are quasiequivalent.*

2°. *Whatever the nuclear spaces* $G_i \in R_i$ $(i = 1, 2, \dots, s)$, *the isomorphism* $E \sim \bigoplus G_i$ *holds if and only if there are* $j_i$, $\sum_i j_i = 0$, *such that* $E_i \sim G_i^{j_i}$ $(i = 1, \dots, s)$.

It remains for us to apply the following result.

**Theorem 5** (see $(^2)$). *Let* $X_i$ *and* $Y_i$ $(i = 1, 2)$ *be linear topological spaces such that every continuous linear mapping from* $X_1$ *to* $Y_2$ *and from* $Y_1$ *to* $X_2$ *is compact. The products* $X_1 \times X_2$ *and* $Y_1 \times Y_2$ *are isomorphic if and only if, for some finite* $j$, $X_1 \sim Y_1^j$ *and* $X_2 \sim Y_2^{-j}$.

An immediate consequence of Theorems 3, 4, and 5 is

**Theorem 6*.** *In the space* $E = \bigoplus E_i$, *where* $E_i \in R_i$ $(i = 1, 2, \dots, s)$–

* This theorem was proved by me jointly with V. P. Zaharyuta.

nuclear Köthe spaces belonging to essentially different (ordered) Riss classes, all bases are quasi-equivalent. Moreover $E \in \mathcal{N}_s$, if $R_i = K(E_i)$ $(i = 1, \dots, s)$.

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*Note: Figure translations are in progress. See original paper for figures.*

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