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A GENERALIZATION OF LYAPUNOV' S SECOND METHOD AND THE STUDY OF SOME RESONANCE PROBLEMS

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The paper gives a generalization of Lyapunov' s second method. The usual requirements imposed on the Lyapunov function and on its derivative are replaced by considerably less restrictive requirements.

The generalized Lyapunov method obtained is applied to the study of resonance problems; problems connected with estimating small denominators in Hamiltonian and more general systems in the "neutral case."

1°. The subject of investigation in the paper is systems of equations

$$\dot{z} = f(t, z) + \mu R(t, z), \tag{1}$$

where the dot denotes differentiation with respect to time t; z, f, and R are vectors with $n + m$ components

$$z = x_1,\ x_2, \dots, x_n,\ y_1, \dots, y_m;$$

μ is a small parameter; $\mu R(t, z)$ are perturbations. We shall assume that, with respect to the variables x, the corresponding unperturbed system

$$\dot{z} = f(t, z) \tag{2}$$

has an equilibrium point, i.e. $f_i(t, 0, y) = 0,\ 1 \le i \le n$. We shall study system (1) for stability with respect to the variables x under

$$|x| \le H, \qquad y \in D, \qquad t \ge 0 \tag{3}$$

(D is the domain of variation of y), where $f(t, z)$ satisfies a Lipschitz condition in z with constant M. We shall also assume that the right-hand sides of systems (1) and (2) in the domain (3) satisfy conditions ensuring the existence of a unique solution of the problem with initial conditions.

The possibility of studying stability only with respect to some of the variables was indicated by A. M. Lyapunov in a note to the work ([1]). Lyapunov' s second method is easily generalized to this case, and its possibilities are broadened if the conditions relating to the Lyapunov function and its derivative are fulfilled uniformly with respect to the variables y from the domain D.

In the paper, in order to study the stability of the equilibrium position of system (2) in the neutral case with respect to the variables x under perturbations μR, it is proposed to use a perturbed Lyapunov function $v = v_0(t, z) + u(t, z, \mu)$. Here v_0 is a Lyapunov function of system (2), and the perturbation u is such that v is not sign-definite and may not have a sign-constant derivative computed along the equations of system (1):

$$\frac{dv}{dt} = \frac{\partial v_0}{\partial t} + \frac{\partial v_0}{\partial z} f + \frac{\partial v_0}{\partial z} \mu R + \frac{\partial u}{\partial t} + \frac{\partial u}{\partial z} f + \frac{\partial u}{\partial z} \mu R = \tag{4}$$

$$= \frac{\partial v_0}{\partial t} + \frac{\partial v_0}{\partial z} f\, \theta(\mu) \varphi$$

where $\theta(\mu) \geq 0$ is a nondecreasing function, and the limit of $\theta(\mu)$ as $\mu \to 0$ is equal to 0. Stability in this case is determined by the smallness of u in some neighborhood of the equilibrium point and by the properties of the function ψ, the mean value of φ, computed along the integral curves of system (2), $z = \bar{z}(t)$,

$$\psi(t_0, z_0, \mu) = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \varphi(t, \bar{z}(t), \mu)\, dt, \qquad \bar{z}(t_0) = z_0. \tag{5}$$

Let us note that in the work ([2]) one condition of Lyapunov' s second method—the sign-definiteness of the derivative—was generalized.

2°. **Theorem 1.** *Let:*

a) *there exist a Lyapunov function $v_0(t, z)$ of system (2), positive definite in x, admitting an infinitely small upper bound with respect to the variables x;*

b) the total derivative v_0, formed by virtue of the equations of system (2), is nonpositive in the domain (3).

Let, for any $\varepsilon > 0$ ($\varepsilon < \varepsilon_0 < H$), one be able to specify a $\gamma > 0$ ($\gamma < \varepsilon$) such that, for all $\eta > \gamma$ ($\eta < \varepsilon$), there exist:

c) $\delta > 0$ such that if $\eta < |x| < \varepsilon$, then uniformly with respect to t_0 and $y_0 \in D$
$$\psi(t_0, x_0, y_0, \mu) < -\delta < 0;$$

d) summable functions $F(t)$ and $M(t)$, constants F_0 and M_0, and also a nondecreasing function $\chi_1(a) > 0$, $\lim_{a\to 0}\chi_1(a) = 0$, such that for $\eta < |x| < \varepsilon$, $t > 0$, and $y \in D$,
$$|\varphi(t, z') - \varphi(t, z'')| < \chi_1(|z' - z''|)F(t); \qquad |R(t, z)| < M(t);$$
$$\int_{t_1}^{t_2} F(t)\,dt \le F_0(t_2 - t_1), \qquad \int_{t_1}^{t_2} M(t)\,dt \le M_0(t_2 - t_1)$$
on any finite interval $[t_1, t_2]$;

e) a nondecreasing function $\chi_2(\mu) > 0$, $\lim_{\mu\to 0}\chi_2(\mu) = 0$, such that
$$|u(t, z, \mu)| \le \chi_2(\mu)$$
for $\eta < |x| < \varepsilon$, $t > 0$, and $y \in D$.

If these conditions are satisfied, one can specify a $\mu_0(\varepsilon)$ such that any solution of system (1), $x = x(t)$, $y = y(t)$, with initial values $x(0) = x_0$ satisfying the condition $|x_0| < \eta$, for $\mu < \mu_0$, for all $t > 0$ satisfies the inequality $|x(t)| < \varepsilon$.

3°. Under cruder assumptions, this theorem was proved in [3], where the function u was constructed in the form of a power series in μ. The circumstance that the conditions of the theorem are imposed not in the disk (3), but in an annulus whose width depends on ε, makes it possible to apply the theorem to the investigation of the stability of various resonance problems and problems connected with the estimation of small divisors.

4°. If condition c) of Theorem 1 is not fulfilled, then stability can be ensured by the requirement $\dot v = 0$; in this case only the condition of positive definiteness of the Lyapunov function is generalized, and the following theorem is valid.

Theorem 2. Let conditions a), b), e) of Theorem 1 be satisfied and let $\varphi(t, z, \mu) = 0$; then the assertion of Theorem 1 is valid.

5°. The apparatus of a perturbed Lyapunov function is also convenient for investigating stability on a finite time interval. In those cases when the sign of the mean ψ is not defined and $\varphi \ne 0$, the following holds.

Theorem 3. Let conditions a), b), and e) of Theorem 1 be satisfied, and let $\varphi(t, z, \mu)$ be bounded for $\eta < |x| < \varepsilon$, $t > 0$, $y \in D$ by a constant φ_0; then one can specify a $\mu_0(\varepsilon)$ such that all solutions whose initial moment satisfies the condition $|x_0| < \eta$, for all $0 < t < T \le \sigma(\varepsilon) \times [2\varphi_0\theta(\mu)]^{-1}$ and $\mu < \mu_0(\varepsilon)$, satisfy the inequality $|x(t)| < \varepsilon$.

Remark. From the estimate given for T it follows that, by choosing sufficiently small μ, one can make $T(\varepsilon, \mu)$ arbitrarily large. Theorem 3 is also valid in the case when φ is an increasing function of time; the estimate of T will then be different.

6°. As an example of application of the theorems formulated above, consider the system

$$\dot{x} = \mu X(x, q), \qquad \dot{q} = \omega(x) + \mu\Phi(x, q), \tag{6}$$

where x and X are s-dimensional, while q, ω, and Φ are m-dimensional vectors. The functions X and Φ are periodic in q with period 2π. A point x_0 is called a resonance point of system (6) if, for an integer vector $n = \{n_1, n_2, \dots, n_m\}$,

$$n\omega(x_0) = 0. \tag{7}$$

The question of the applicability of the averaging method to systems of the form (6) in the presence of resonances was considered in [4] (in the two-frequency case). In [5, 6] the stability of resonance points of system (6) was investigated in a certain special sense. Our aim is to

investigate the resonance points of system (6) for Lyapunov stability with respect to the variables x.

Let x_0 be an equilibrium position of the system obtained by the formal averaging of the functions $X(x, q)$ with respect to the angles q,

$$\dot{x} = \mu X_0(x), \qquad X_0(x_0) = 0. \tag{8}$$

Suppose that, for system (8), the conditions of Lyapunov' s theorem on asymptotic stability with the function $v_0(x)$ are satisfied. Conditions a) and b) of Theorem 1 are then satisfied, since (6) is a system of the form (2), where $f_i \equiv 0$, $1 \leq i \leq n$. First consider the case when at the point x_0 only one combination frequency $n\omega = \lambda$ vanishes and has a strict extremum at x_0. In the functions $X(x, q)$ separate a finite sum $\widetilde{X}$ and a remainder R_N

$$X(x, q) = X_0(x) + \sum_{k=1}^{N} X_k(x) e^{i(kq)} + \sum_{k>N}^{\infty} X_k(x) e^{i(kq)} = X_0(x) + \widetilde{X} + R_N. \tag{9}$$

Having fixed a number $\varepsilon > 0$, choose $N(\varepsilon)$ so that all terms with resonances lying closer than 2ε to x_0 enter the remainder R_N. Then $\widetilde{X}$ contains only one resonance frequency λ, $\lambda(x_0) = 0$, which outside the η-neighborhood of x_0 is different from 0. Let $v = v_0(x)$, i.e. $u \equiv 0$; then

$$\dot{v} = \mu\varphi = \mu\nabla v_0(X_0 + R_N) + \mu\nabla v_0\widetilde{X}. \tag{10}$$

Impose a restriction relating the rate of growth of $X_0(x)$ to the rate of convergence of the series $X(x, q)$, requiring that in a neighborhood of x_0, $X_0(x)$

increase faster than $R_{N(\varepsilon)}$. Then one can choose ε_0 so that condition c) of Theorem 1 is fulfilled for $\varepsilon < \varepsilon_0$. Suppose also that the smoothness conditions d) are satisfied. Thus, all the conditions of Theorem 1 are fulfilled; consequently, the resonance point x_0 is Lyapunov stable.

7°. Suppose equation (7) determines a certain resonance curve (surface). From the sum $\widetilde{X}$ separate the terms $\overline{X}(x, Q)$, where $Q = nq$. $\overline{X}$ does not oscillate on the resonance curve (7). Analogously, in the case of several combination frequencies λ, include in $\overline{X}$ the terms containing slowly varying angles Q. To satisfy condition c) of Theorem 1, require that $\overline{X}(x, Q)$ vanish for $x = x_0$ and grow more slowly than $X_0(x)$.

8°. If the curves $X_0(x) = 0$ (or $X_{0i}(x) \equiv 0$) coincide with the resonance curve (surface) $n\omega(x) = 0$, then it is natural to investigate for stability this curve, rather than a point. For this purpose it is expedient to pass to new coordinates associated with the resonance curve, one of which is measured along the normal to the curve, and then to carry out the stability investigation with respect to this coordinate.

9°. Suppose that, for system (8), the conditions of Lyapunov' s theorem on stability with the function $v_0(x)$ are satisfied, i.e. $\nabla v_0 X_0(x) \le 0$. Suppose that the point x_0 is an isolated resonance point, as defined in item 6°. For $\varepsilon > 0$ introduce $\widetilde{X}$ and $R_{N(\varepsilon)}$, just as in item 6°. Let $u = \mu U$, where U is a solution, bounded in the annulus $\eta < |x| < \varepsilon$, of the equation

$$\frac{\partial U}{\partial q}\,\omega(x) = -\frac{\partial v_0}{\partial x}\,\widetilde{X}(x, q). \tag{11}$$

In this case

$$\dot{v} = \mu\left[\frac{\partial v_0}{\partial x} R_N + \mu\frac{\partial U}{\partial x}(X_0 + \widetilde{X} + R_N) + \mu\frac{\partial U}{\partial q}\Phi\right] \equiv \mu\varphi(x, q, \mu, \varepsilon).$$

Negative definiteness of the average of φ makes it possible to apply Theorem 1 in the first approximation, while positive definiteness makes it possible to detect instability. If, however, φ is bounded in modulus, then, applying Theorem 3, we obtain an estimate for a finite interval on which the solution in the variable x does not leave the ε-neighborhood of the point x_0.

10°. Theorems 2 and 3 make it possible to investigate, for stability with respect to the variables p, a system with Hamiltonian function

$$H(p, q) = H_0 + \mu H_1 = H_0(p) + \mu\sum_k h_k(p) e^{i(kq)}. \tag{12}$$

The corresponding system of equations is a particular case of system (6). In particular, here p and q have the same dimension m, and $X_0(x) \equiv 0$. Therefore everything said in §§ 8 and 9 applies to the system with Hamiltonian function (12).

Let us dwell on the case when $H(p, q)$ depends on one phase $\psi = k_0 q$,

$$H_1(p, q) = H_1(p, \psi) = \sum_{n=0}^{\infty} h_{nk_0} e^{in\psi} + h_{-nk_0} e^{-in\psi}.$$

Introduce the notation $k_0 \partial H_0/\partial p = \omega_0$, $k_0 \partial h_0/\partial p = \Omega_0$, and require uniform convergence of the series in the right-hand sides of the equations

$$\dot{p} = -\mu k_0 \partial H_1/\partial \psi, \qquad \dot{\psi} = \omega_0 + \mu \Omega_0 + \mu k_0 \partial H_1/\partial p. \tag{13}$$

It is evident from this that the variables p vary in the direction of the vector k_0, so that in this direction it is convenient to introduce the variable x, measured from the point p_0 being investigated for stability. Consider the most interesting case, when the point p_0 lies on the resonance curve (surface) $\omega_0(p) = 0$. Require that the angle α formed by the vector k_0 with the tangent to the resonance curve be everywhere strictly greater than 0: $0 < \alpha_0 < \alpha$. At the same time let it also be the case that for every $\eta > 0$ there exists a $\delta > 0$ such that if $\eta < |x|$, then $\delta < |\omega(x)|$. As $v_0(p)$ choose $v_0(p) = x^2$, and define the function u by means of a power series in μ

$$u = \sum_{s=1}^{\infty} \mu^s v_s(p, \psi), \tag{14}$$

where v_s is a bounded solution of the recurrent system of equations

$$\frac{\partial v_1}{\partial \psi} \omega_0 = k_0 \frac{\partial v_0}{\partial p} \frac{\partial H_1}{\partial \psi}, \dots$$

$$\dots, \quad \frac{\partial v_s}{\partial \psi} \omega_0 = \left(k_0 \frac{\partial v_{s-1}}{\partial p} \right) \frac{\partial H_1}{\partial \psi} - \left(k_0 \frac{\partial H_1}{\partial p} \right) \frac{\partial v_{s-1}}{\partial \psi} - \Omega_0 \frac{\partial v_{s-1}}{\partial \psi}. \tag{15}$$

Here $\varphi = 0$. The functions v_s are obtained in the form of polynomials of degree s with respect to H_1. It is easy to show that for sufficiently small μ and for $\eta < |x|$ the series (14) converges, and thereby the conditions of Theorem 2 are satisfied. The estimates improve if the point p_0 lies to the side of the line of resonance.

Remark. If the indicated restriction on the angle α is not introduced, stability may fail even in the two-dimensional case. This can be shown on the two-dimensional example of a system with Hamiltonian function

$$H(p,q) = \frac{1}{2}[p_1^2 - p_2^2] + \mu A \sin(q_1 + q_2), \tag{16}$$

where A is a constant. Hamilton' s equations have the form

$$\dot{p}_1 = -\mu A \cos\psi, \qquad \dot{p}_2 = -\mu A \cos\psi, \qquad \dot{\psi} = p_1 - p_2. \tag{17}$$

Here $\psi = q_1 + q_2$. The resonance line is the straight line $p_1 = p_2$; along it the change of p_1 and p_2 takes place. Therefore, if the initial point is chosen on the resonance line, all trajectories will lie on it; in this case ψ is preserved, while p, for values of ψ for which $\cos\psi \neq 0$, tends to ∞.

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Note: Figure translations are in progress. See original paper for figures.