ON THE NATURE OF THE SPECTRUM OF A SELF-ADJOINT EXTENSION OF THE LAPLACE OPERATOR IN A BOUNDED DOMAIN

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Abstract

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ON THE NATURE OF THE SPECTRUM OF A SELF-ADJOINT EXTENSION OF THE LAPLACE OPERATOR IN A BOUNDED DOMAIN

(FUNDAMENTAL SYSTEMS OF FUNCTIONS WITH AN ARBITRARY PREASSIGNED SUBSEQUENCE OF FUNDAMENTAL NUMBERS)

(Presented by Academician A. N. Tikhonov on 28 XI 1969)

As in the papers ([1,2]), we shall call a complete orthonormal system $\{u_k(x)\}$ in an arbitrary N-dimensional domain G a fundamental system of functions (f.s.f.) of the Laplace operator in this domain if each function $u_k(x)$ belongs, inside G, to the class C^2 and, for some nonnegative number λ_k, satisfies inside G the equation $\Delta u_k + \lambda_k u_k = 0$. We shall call the numbers λ_k fundamental numbers.

In the present paper, for a bounded domain G, the question is studied of the nature of the distribution of the fundamental numbers on the number axis and of the existence of finite points of accumulation for these numbers. Since each f.s.f. of the Laplace operator in a domain G corresponds to some self-adjoint nonnegative extension of the minimal operator generated in $L_2(G)$ by the operator $-\Delta$, what is in fact being studied is the nature of the spectrum of self-adjoint nonnegative extensions of the Laplace operator in a bounded domain G. We shall establish that for a domain of two or more dimensions the nature of the indicated spectrum turns out to be much more complicated than for those self-adjoint extensions of elliptic operators that have been studied in numerous works on the asymptotics of eigenvalues (beginning with the old works of H. Weyl ([3]) and T. Carleman ([4]) and ending with the recent works of S. Mizohata ([5]) and S. Agmon ([6])). We also note that the f.s.f. of the Laplace operator studied below do not belong to the class of those self-adjoint extensions of elliptic operators for which M. I. Vishik ([7]) established the theorem on normal solvability. Nevertheless, all the results established in ([1,2]) are valid for the f.s.f. studied by us.

We pass to the precise formulation of the results.

1°. For completeness in clarifying the question, we begin with the one-dimensional case. In this case the only connected bounded domain is an interval.

Theorem 1. *The fundamental numbers λ_n of an arbitrary f.s.f. of the operator $\Delta = d^2/dx^2$ on the interval $0 \le x \le l$ have no finite points of accumulation and, when numbered in increasing order, satisfy the inequalities*

$$An^2 \le \lambda_n \le Bn^2 \qquad (B \ge A > 0). \tag{1}$$

The very fact that finite points of accumulation are absent is trivial and follows, for example, from Theorem 5 on p. 212 of the monograph ([8]) of M. A. Naimark; however, wishing to establish the two-sided estimates (1), we indicate a scheme for the proof of Theorem 1. The basic characteristic property of every f.s.f. is the mean-value formula, which in the case under consideration

in this case has the form

$$u_\lambda(x+h) + u_\lambda(x-h) = 2u_\lambda(x)\cos h\sqrt{\lambda}. \tag{2}$$

(Here $u_\lambda(x)$ is the fundamental function corresponding to the fundamental number λ; the point x and the number h are such that the segment $[x-h, x+h]$ is contained in the interval $(0, l)$.)

Relying on formula (2) and repeating the scheme of reasoning of item 3, § 1, Ch. 2 of [1], we establish for the sum of squares of the fundamental functions the estimate

$$\sum_{\mu < \sqrt{\lambda_n} < \mu+1} u_n^2(x) = O(1), \tag{3}$$

uniformly in every strictly interior subinterval of the segment $[0, l]$ and, in particular, on the segment $[l/4, 3l/4]$. But from the uniformity of estimate (3) on $[l/4, 3l/4]$ and from the same formula (2) it follows at once that estimate (3) is uniform on the whole interval $0 < x < l$. Then, integrating estimate (3) over $(0, l)$, we establish the left inequality (1), while the validity of the right inequality (1) follows from the main theorem of [2].

Remark 1. Theorem 1 remains valid if, instead of the segment $[0, l]$, one takes a bounded one-dimensional set G consisting of a finite number of nonintersecting intervals. But if a bounded one-dimensional set G consists of an infinite number of nonintersecting intervals (for example, $G = (1/2, 1) \cup (1/4, 1/2) \cup (1/8, 1/4) \cup \cdots$), then the method given below makes it possible to construct on such a set an f.s.f. for which a previously prescribed sequence of positive numbers serves as a subsequence of its fundamental numbers.*

2°. We turn to the study of the spectrum of the f.s.f. of the Laplace operator in a bounded domain of two and a greater number of dimensions.

Theorem 2. *Let $\{\lambda_n^0\}$ be an arbitrary sequence of numbers satisfying the condition $0 \le \lambda_n^0 < n^2$. Then there exists an f.s.f. of the Laplace operator in the square $Q = [0 \le x \le \pi] \times [0 \le y \le \pi]$ for which the sequence $\{\lambda_n^0\}$ is a subsequence of the fundamental numbers.*

Proof. For each number $n = 1, 2, 3, ...$, consider the eigenfunction problem

$$X''(x) + \mu X(x) = 0 \qquad (0 < x < \pi), \tag{4}$$

$$X(0) = 0, \qquad X'(\pi) = h_{nX}(\pi), \qquad h_n = \sqrt{n^2 - \lambda_n^0}\, \operatorname{cth} \pi \sqrt{n^2 - \lambda_n^0}\,.$$

Since the equation $X'' + \mu X = 0$ and the condition $X(0) = 0$ are satisfied by the functions $\operatorname{sh} x\sqrt{-\mu}$ $(\mu < 0)$, $\sin x\sqrt{\mu}$ $(\mu > 0)$, and x $(\mu = 0)$, it follows from the boundary condition at $x = \pi$ that the eigenvalues are the numbers $\mu_n^0 = \lambda_n^0 - n^2 < 0$ and μ_n^k, $k = 1, 2, ...$, where $\nu = \sqrt{\mu_n^k}$ are the positive roots, arranged in increasing order, of the equation $\nu \operatorname{ctg} \pi\nu = h_n$. Consequently,

$$k^2 < \mu_n^k < (k+1)^2 \qquad (n = 1, 2, ... ;\ k = 1, 2, ...). \tag{5}$$

The eigenfunctions are

$$X_n^0(x) = N_n^0 \operatorname{sh} x\sqrt{-\mu_n^0}, \qquad X_n^k(x) = N_n^k \sin x\sqrt{\mu_n^k}, \qquad k = 1, 2, ...$$

Here N_n^k are normalizing factors. For each n this system of eigenfunctions is orthogonal and complete, since problem (4) is self-adjoint–

* This means that the fundamental numbers may have limit points covering the entire positive half-axis or ar

finite. Hence it follows that the set of functions

$$u_n^k(x, y) = X_n^k(x)\pi^{-1/2} \sin ny \qquad (n = 1, 2, ... ;\ k = 0, 1, 2, ...) \tag{6}$$

forms an F.S.F. of the Laplace operator in the domain Q, while the set of fundamental numbers $\lambda_n^k = \mu_n^k + n^2$ $(n = 1, 2, ... ;\ k = 0, 1, 2, ...)$ contains the prescribed subsequence $\{\lambda_n^0\}$.

Remark 2. If from the set of all numbers λ_n^k one removes the initially prescribed subsequence $\{\lambda_n^0\}$, then the remaining numbers λ_n^k $(n \geqslant 1,\ k \geqslant 1)$, by virtue of

(5), will have the same law of asymptotic distribution as the eigenvalues of the first boundary-value problem for the square.

Remark 3. Instead of a square one may consider a disk. Taking an arbitrary sequence of positive numbers $\{\lambda_n^0\}$ and considering, instead of the boundary-value problem (4), the corresponding boundary-value problem for the Bessel equation of order n, we construct an F.S.F. of the Laplace operator in the disk for which $\{\lambda_n^0\}$ serves as a subsequence of fundamental numbers.

Remark 4. Analogously, for any prescribed sequence of positive numbers $\{\lambda_n^0\}$, one constructs an F.S.F. of the Laplace operator in an N-dimensional cylinder B, equal to the product of the interval $[0 \leqslant x_1 \leqslant \pi]$ by an $(N-1)$-dimensional bounded domain $\bar{G}$, so that the sequence $\{\lambda_n^0\}$ is a subsequence of the fundamental numbers for this F.S.F. (It is enough in the arguments of Theorem 2 to replace n^2 and $\pi^{-1/2}\sin ny$, respectively, by the eigenvalues $\bar{\lambda}_n$ and eigenfunctions $\bar{v}_n(x_2, \dots, x_N)$ of the domain $\bar{G}$.)

Corollary 1. *In a bounded domain of $N \geqslant 2$ dimensions there exists an F.S.F. of the Laplace operator whose fundamental numbers have either isolated points of condensation or points of condensation covering the entire half-line $\lambda \geqslant 0$, or some closed part of this half-line.*

Corollary 2. *Whatever increasing function $f(\lambda)$ on the half-line $\lambda \geqslant 0$ may be, there exists a self-adjoint extension of the Laplace operator in a bounded domain of $N \geqslant 2$ dimensions whose spectrum is discrete, and for the number $n(\lambda)$ of eigenvalues not exceeding λ the equality $n(\lambda) = f(\lambda) + O(\lambda^{N/2})$ holds (see Remark 2).*

Remark 5. If in Theorem 2 one drops the condition $0 \leqslant \lambda_n^0 < n^2$, then the method used in its proof permits one to construct a complete orthonormal system of functions of the form (6), which are eigenfunctions of the Laplace operator in the domain Q, but now not all eigenvalues $\lambda_n^k = \mu_n^k + n^2$ may be positive.

Remark 6. Let us indicate the boundary conditions satisfied by the constructed eigenfunctions (6). Obviously, $u_n^k(x,0) = u_n^k(x,\pi) = u_n^k(0,y) = 0$, and by virtue of (4) $\partial u/\partial x = Hu$ for $x = \pi$, where H is the self-adjoint operator which assigns to a function $f(y) = \sum b_n \sin ny$ the function $Hf(y) = \sum h_n b_n \sin ny$. It is not difficult to express the operator H explicitly in terms of a certain integral operator.

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Note: Figure translations are in progress. See original paper for figures.