UPPER BOUNDS OF BEST APPROXIMATIONS ON CLASSES OF DIFFERENTIABLE PERIODIC FUNCTIONS IN THE METRICS \(C\) AND \(L\)

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Abstract

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UPPER BOUNDS OF BEST APPROXIMATIONS ON CLASSES OF DIFFERENTIABLE PERIODIC FUNCTIONS IN THE METRICS C AND L

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Let $C_{2\pi}$ be the set of functions continuous on the whole axis with period 2π, let X be the metric C or L, and

$$E_n(f)_X = \inf_{T_n} \|f - T_n\|_X \qquad (n = 0, 1, 2, ...)$$

be the best approximation of a function $f \in C_{2\pi}$ by trigonometric polynomials $T_n(x)$ of order $\leq n$. We consider the problem of finding the exact upper bound

$$E_n(\mathfrak{M})_X = \sup_{f \in \mathfrak{M}} E_n(f)_X,$$

where $\mathfrak{M}$ is a certain class of functions.

Denote by $W^r H_\omega$ $(r = 0, 1, 2, ...;\ W^0 H_\omega = H_\omega)$ the class of r-times continuously differentiable functions $f(x) \in C_{2\pi}$ for which

$$|f^{(r)}(x') - f^{(r)}(x'')| \leq \omega(|x' - x''|) \qquad (f^{(0)} = f),$$

where $\omega(t)$ is a given modulus of continuity. For $\omega(t) = Kt^\alpha$ $(0 \leq t \leq \pi,\ 0 < \alpha \leq 1)$ we shall write $W^r K H^\alpha$. Note that the class $W^{r-1} K H^1$ for $r = 1, 2, ...$ coincides with the class W^{rK} of functions $f \in C_{2\pi}$ for which the $(r-1)$-st derivative is absolutely continuous and $|f^{(r)}(x)| \leq K$ almost everywhere.

For a given modulus of continuity $\omega(t)$ and all $n = 1, 2, ...$ and $r = 0, 1, 2, ...$, consider the functions $f_{nr}(\omega, x)$ of period $2\pi/n$ with mean value over a period equal to zero, whose derivative of order r is odd and is defined by the equalities

$$f_{nr}^{(r)}(\omega, x) = \begin{cases} \frac{1}{2}\,\omega(2x), & 0 \le x \le \pi/2n, \\ \frac{1}{2}\,\omega(2\pi/n - 2x), & \pi/2n \le x \le \pi/n. \end{cases}$$

It is easy to verify that if $\omega(t)$ is a convex upward modulus of continuity, then $f_{nr}(\omega, x) \in W^r H_\omega$.

The principal content of the present note is the following.

Theorem 1. *Whatever convex upward modulus of continuity $\omega(t)$ may be, for all $r = 0, 1, 2, \ldots$ the equalities*

$$E_{n-1}(W^r H_\omega)_X = E_{n-1}(f_{nr}(\omega, x))_X = \|f_{nr}(\omega, x)\|_X \qquad (n = 1, 2, \ldots), \tag{1}$$

hold, where X is the metric C or L.

We note that, in a number of special cases, the values of the upper bounds $E_n(W^r H_\omega)_X$ were obtained earlier. It was proved by J. Favard ([1]), and also by N. I. Akhiezer and M. G. Krein ([2]), that

$$E_{n-1}(W^r K H^1)_C = E_{n-1}(W^{r+1} K)_C = \frac{4K}{\pi n^{r+1}} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu r}}{(2\nu+1)^{r+2}} \tag{2}$$

$$(r = 0, 1, 2, \ldots;\ n = 1, 2, \ldots).$$

Best approximation in the mean on classes of functions was first considered in a work of S. M. Nikol' skii ([3]), where, by means of general theo-

···concerning approximation in a Banach space, a number of definitive results in the metric L have been obtained. Exact estimates of approximation in the mean are contained in the papers ([4,5]) and ([6]), where the values of $E_n(W^{rK})_L$ were computed $(r = 1, 2, \ldots)$.

For an arbitrary convex modulus of continuity $\omega(t)$, the values of the upper bounds $E_n(W^r H_\omega)_C$ for $0 \le r \le 3$ are given in ([7–10]), and the proof was based on exact estimates, found by the author, of intermediate approximation of functions of the class $W^r H_\omega$ by functions from $W^r K H^1$, on the optimal choice of the constant K, and on equalities (2).

It was natural, however, to try to solve the problem of computing the upper bounds $E_n(W^r H_\omega)_C$ and $E_n(W^r H_\omega)_L$ by a "direct" method, i.e., using only properties of best approximations in the metrics C and L. The author found a proof of the equalities (1) for $X = C$ and L and for all $r = 0, 1, 2, \ldots$, based on the relations obtained by S. M. Nikol' skii ([3]):

$$E_{n-1}(f)_C = \sup_{h \in H_L^n} \int_0^{2\pi} f(t)h(t)\,dt, \qquad E_{n-1}(f)_L = \sup_{h \in H_M^n} \int_0^{2\pi} f(t)h(t)\,dt, \quad (3)$$

where H_L^n (H_M^n) are the sets of functions $h(t)$ of period 2π, orthogonal to all trigonometric polynomials of degree $n-1$ and such that $\|h\|_L \leq 1$ ($\|h\|_M = \sup_x |h(x)| \leq 1$). An essential role here was played by the investigation—of independent interest—of the properties of functions that are periodic integrals of $h(t)$, i.e., of the form

$$g(x) = \frac{1}{\pi} \int_0^{2\pi} \sum_{k=1}^{\infty} \frac{\cos[k(t-x) + r\pi/2]}{k^r} h(t)\,dt \qquad (r = 1, 2, \ldots), \qquad (4)$$

where $h \in H_L^n$ or $h \in H_M^n$. (The sets of such functions will be denoted respectively by $W^r H_L^n$ and $W^r H_M^n$.) In order to formulate the results obtained in this direction, we introduce the following definition.

We shall call a function $\varphi(x)$, continuous on the whole axis, **simple** if it is equal to zero outside some interval (α, β), $|\varphi(x)| > 0$ for $x \in (\alpha, \beta)$, and for every y, $0 < y < \max_x |\varphi(x)|$, the equation $|\varphi(x)| - y = 0$ has exactly two roots on (α, β). If $g \in C^1_{2\pi}$ (i.e., $g \in C_{2\pi}$ and has a continuous first derivative), and $g(x_0) = 0$, but $g \neq 0$, then $g(x)$ can be represented on $[x_0, x_0 + 2\pi]$ in the form of a finite or countable sum

$$g(x) = \sum_k \varphi_k(x) \qquad (x_0 \leq x \leq x_0 + 2\pi),$$

where φ_k are simple functions such that $|\varphi_k(x)| > 0$ on the interval $(\alpha_k, \beta_k) \subset (x_0, x_0 + 2\pi)$, $\varphi_k = 0$ outside (α_k, β_k), and

$$\sum_k \|\varphi_k\|_C = \frac{1}{2} \bigvee_0^{2\pi}(g), \qquad \sum_k \|\varphi_k\|_L = \|g\|_L.$$

Let $\bar{\varphi}_k(x)$ be the rearrangement of the functions $|\varphi_k(x)|$ in decreasing order, i.e., the function inverse to $x = M(y)$, where $M(y)$ is the measure of the set on which $|\varphi_k(x)| \geq y$ (see, for example, ([11]), p. 332). We shall put $\bar{\varphi}_k(x) = 0$ for $x > \beta_k - \alpha_k$. Denoting

$$\Phi(g, x) = \sum_k \bar{\varphi}_k(x) \qquad (0 \leq x \leq 2\pi),$$

to each function $g(x)$ we uniquely assign a function $\Phi(g, x)$, moreover

$$\Phi(g(t+a), x) = \Phi(g(t), x)$$

for any a.

Of importance in the study of the properties of functions of the form (4) is

Theorem 2. If $g \in C^3_{2\pi}$, $g(0) = 0$, then

$$|\Phi'(g, x)| \leq \frac{1}{4} \int_0^x \Phi(g'', t)\, dt \tag{5}$$

everywhere on $(0, 2\pi)$, where $\Phi'(g, x)$ exists. The inequality is sharp on the set $C^3_{2\pi}$.

Let, for each $n = 1, 2, ...$, for $0 \leq x \leq \pi/n$,

$$\Phi_{n1}(x) = \frac{1}{2}, \qquad \Phi_{nr}(x) = \frac{1}{2} \int_0^{\pi/n - x} \Phi_{n,r-1}(t)\, dt \quad (r = 2, 3, ...).$$

It is easy to verify that for $r \geq 2$

$$\Phi_{nr}(x) = \Phi(g_{nr}, x) \quad (0 \leq x \leq \pi/n),$$

where $g_{nr}(x)$ is the $(r-1)$-st periodic integral of the function

$$g_{n1}(x) = \frac{1}{4n} \operatorname{sign} \sin nx.$$

With the aid of inequality (5) one proves

Theorem 3. If $g \in W^r H^n_L$ $(g \in W^r H^n_M)$ $(n = 1, 2, ...;\ r = 2, 3, ...)$, then $\Phi'(g, x) \geq \Phi'_{nr}(x)$ (respectively $\Phi'(g, x) \geq 4n\Phi_{n,r+1}(x)$) at all points of the interval $(0, \pi/n)$ at which $\Phi'(g, x)$ exists.

Theorem 2 makes it possible to obtain an exact upper estimate for the integral

$$\int_0^{2\pi} f(t) g(t)\, dt,$$

where $f \in H_\omega$, and $g \in W^r H^n_L$ or $g \in W^r H^n_M$, for $r = 1, 2, ...$

In the case $r = 0$, an analogous result is obtained by passing to Steklov functions. Taking into account relations (3), this gives Theorem 1. We note that the values of the upper bounds $E_{n-1}(W^r H_\omega)_X$ can be written with the aid of the functions Φ_{nr}:

$$E_{n-1}(H_\omega)_C = \Phi_{n1}\left(\frac{\pi}{n}\right) \omega\left(\frac{\pi}{n}\right),$$

$$E_{n-1}(W^r H_\omega)_C = \frac{1}{2} \int_0^{\pi/n} \omega(t) \Phi_{nr}\left(\frac{\pi}{n} - t\right) dt \quad (r = 1, 2, ...),$$

$$E_{n-1}(W^r H_\omega)_L = 2n \int_0^{\pi/n} \omega(t) \Phi_{n,r+1}\left(\frac{\pi}{n} - t\right) dt \quad (r = 0, 1, ...).$$

In conclusion we note that equalities (1) give upper estimates for the widths of the classes $W^r H_\omega$ in the spaces of continuous and summable functions. (In the case $\omega(t) = Kt$, $r = 0, 1, 2, ...$, and for convex $\omega(t)$ with $r \leq 3$, the widths of these classes in the space of continuous functions with the uniform metric were computed in ([12], [10]).)

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Note: Figure translations are in progress. See original paper for figures.