CONTINUOUS AND SEMICONTINUOUS o-METRICS

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Abstract

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CONTINUOUS AND SEMICONTINUOUS o-METRICS

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Definition 1 (S. I. Nedev ([1])). A topological space X is called **o-metrizable** by an **o-metric** ρ if ρ is a nonnegative function on the square $X \times X$ satisfying the following two conditions: 1) $\rho(x, y) = 0$ if and only if $x = y$; 2) if $F \subset X$, then F is closed in X if and only if for every point $x \notin F$ the relation

$$\rho(x, F) = \inf\{\rho(x, y)/y \in F\} > 0.$$

holds.

If the function ρ is symmetric, i.e., if for any two points $x, y \in X$ the equality $\rho(x, y) = \rho(y, x)$ holds, then ρ is called **symmetric**, and the space X is called **symmetrizable**.

If, in addition to conditions 1) and 2), the function ρ also satisfies the triangle inequality, i.e., if for any three points $x, y, z \in X$ the inequality $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ holds, then ρ is called a **Δ-metric**, and the space X is called **Δ-metrizable**.

If the function ρ is both symmetric and a Δ-metric, then it is called a **metric**, and the space X is called **metrizable**.

The class of o-metrizable spaces is sufficiently broad—all spaces with the first axiom of countability are o-metrizable. There is, however, the possibility, by means of purely analytic restrictions on the o-metric, of carrying out a very fine classification of that extensive domain of topological spaces filled by o-metrizable spaces. Here we shall give only those restrictions that we shall need in this paper.

Condition (h). For any point $x \in X$ and for any $\varepsilon > 0$, $x \in \langle O^\rho_\varepsilon x \rangle$, where $O^\rho_\varepsilon x = \{y/\rho(x, y) < \varepsilon\}$.

An o-metric ρ satisfying condition (h) will be called a **strong o-metric**, and the space X whose topology it determines will be called **strongly o-metrizable** by the o-metric ρ.

Condition (AH) (P. S. Aleksandrov, V. V. Nemytzkiĭ ([2])). For any point $x \in X$ and any $\varepsilon > 0$ there exists $\delta = \delta(x, \varepsilon) > 0$ such that from $\rho(x, y) < \delta$ and $\rho(x, z) < \delta$ it follows that $\rho(y, z) < \varepsilon$.

Weak Cauchy condition (A. V. Arkhangel' skiĭ ([3])). If the set A is not closed, then for any $\varepsilon > 0$ there exist points $x, y \in A$ such that $\rho(x, y) < \varepsilon$ and $x \neq y$.

Condition (K) (A. V. Arkhangel' skiĭ ([4])). If F and Φ are bicompact, then $\rho(F, \Phi) > 0$.

Condition (A) (A. V. Arkhangel' skiĭ ([4])). If the set F is closed, and Φ is bicompact, then $\rho(F, \Phi) > 0$.

Theorem 1 (A. V. Arkhangel' skiĭ ([4])). *If a T_2-space X is symmetrizable by a symmetric satisfying condition (A), then X is metrizable.*

Theorem 2 (P. S. Aleksandrov, V. V. Nemytzkiĭ ([2])). *A T_1-space X is symmetrizable by a symmetric satisfying*

conditions (h) and (AH), when X has a refining sequence of covers*.

Theorem 3 (C. I. Nedev (5), Theorem 31**). If the space X is o-metrizable by an o-metric ρ and, for each point $x_0 \in X$, the function $f_{x_0}(x) = \rho(x_0, x)$ is uniformly continuous with respect to ρ, then X is metrizable.

Definition 2. If a space X is o-metrizable by an o-metric ρ and the mapping $\rho : X \times X \to E^1$ (E^1 is the space of real numbers) is continuous, then the o-metric ρ is called a **continuous o-metric**.

Definition 3. If a space X is o-metrizable by an o-metric ρ and if, for each point $x_0 \in X$, the function $f_{x_0}(x) = \rho(x_0, x)$ is continuous on X, then ρ is called a **weakly continuous o-metric**.

Lemma 1. An o-metric ρ generating the topology of a space X is a strong o-metric if and only if, for each point $x_0 \in X$, the function $f_{x_0}(x) = \rho(x_0, x)$ is continuous at the point x_0.

Lemma 2. An o-metric ρ generating the topology of a space X is a strong o-metric satisfying condition (AH) if and only if the mapping $\rho : X \times X \to E^1$ is continuous at points of the diagonal.

Lemma 3. If a space X is o-metrizable by a lower semicontinuous o-metric ρ, then X is completely regular and all ball neighborhoods of points with respect to ρ (i.e., sets of the form $O^{\rho}_{\varepsilon} x$) are open in X.

Lemma 4. Let a space X be symmetrized by a symmetric ρ. If all ball neighborhoods of points with respect to ρ are open in X, then ρ satisfies the weak Cauchy condition.

Theorem 4. If a space X is o-metrizable by a continuous o-metric ρ, then X is completely regular, and ρ is a strong o-metric satisfying conditions (AH) and (K). Consequently, X has a refining sequence of covers.

Example 1 (A. V. Arhangel' skii). An example of a finally compact nonmetrizable space X, symmetrized by a lower semicontinuous symmetric.

The points of the space X are the points of the upper half-plane E and the points of the boundary line E^1, i.e., $X = E \cup E^1$. For any two points $x, y \in X$, let $|x-y|$ denote the ordinary Euclidean distance between them, and let $a(x, y)$ be the least angle between the lines (x, y) and E^1, in radians. Introduce on X the symmetric $d(x, y)$ as follows: a) if $x \in E^1$, then $d(x, y) = d(y, x) = |x-y| + a(x, y)$; b) if $x, y \in E$ and x and y lie on the same horizontal line, then $d(x, y) = d(y, x) = |x - y|$; c) if $x, y \in E$, y is above x, and z is the common point of the lines (x, y) and E^1, then

$$d(x, y) = d(y, x) = |x - y|d(y, z)/|y - z|.$$

The lower semicontinuity of the symmetric d is checked directly. It is also easy to note that both E and E^1 inherit from X their usual Euclidean topologies. It follows that X is finally compact. On the other hand, the weight of the space X is not less than the continuum and, hence, X is not metrizable.

Corollary 1. Condition (AH) and the weak Cauchy condition are not equivalent.

Example 2. An example of a space X without a σ-conservative network***, symmetrized by a lower semicontinuous symmetric.

The points of the space X are the points of the plane. Introduce on X the symmetric

$$d(x, y) = |x - y| + a(x, y)$$

(the notation is taken from Example 1).

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* A sequence of covers $\{\omega_n \mid n = 1, 2, ...\}$ is called refining if for every point x and every neighborhood O_x of it there is an n such that $\omega_n x \subset O_x$ (P. S. Aleksandrov, P. S. Urysohn (6)).

** Theorem 31 from (5) is considerably stronger than Theorem 3 given here.

*** A family $S = \{s_\alpha \mid \alpha \in A\}$ of subsets of a space X is called a network if, for every open set U in X and every point $x \in U$, there is an $\alpha \in A$ such that $x \in s_\alpha \subset U$ (A. V. Arhangel' skii (8)).

The first example of a regular symmetrizable space without a σ-conservative network was constructed by J. A. Kofner (7). We note that every space o-metrizable by a continuous o-metric has a σ-discrete network.

Theorem 5. Let X be symmetrizable by a semicontinuous symmetric, and let

$$\tau = \inf\{|X'|/X' \subset X,\ X' \text{ everywhere dense in } X\}.$$

Then X admits a one-to-one continuous mapping into I^τ (I is the unit interval).

Corollary 2. If a separable space X is symmetrizable by a semicontinuous symmetric, then X is condensed onto a metrizable space, and this condensation is a homeomorphism on an everywhere dense set and, consequently, X contains an everywhere dense metrizable subspace which is a set of type G_δ in X.

Theorem 6. If a space X is symmetrizable by such a symmetric ρ that, for every countable closed set $F \subset X$, the function $f_F(x) = \rho(F, x)$ is continuous on X, then X is metrizable, since then ρ satisfies condition (A).

We note that in Theorem 6 the words "symmetrizable" and "symmetric" may be replaced by the words "Δ-metrizable" and "Δ-metric."

Lemma 5. If a space X is symmetrizable by a continuous symmetric ρ, then, for every bicompact $F \subset X$, the function $f_F(x) = \rho(F, x)$ is continuous on X.

Example 3. An example of a nonmetrizable space X, symmetrizable by a continuous symmetric, representable as the sum $X = X_1 \cup X_2$, where X_1 is countable and X_2 is discrete in X.

Let $X' = E \cup E^1$. For each point $x \in X'$ let h_x denote the ordinate, and l_x the abscissa, of the point x (in some fixed orthogonal coordinate system having the line E^1 as the axis of abscissas). Introduce on X' a symmetric $d(x, y)$ by the following rule: a) if $h = \max\{h_x, h_y\} > 0$, then

$$d(x, y) = |l_x - l_y| + |x - y|;$$

b) if $h = 0$ and $x \neq y$, then

$$d(x, y) = 1 + |x - y|;$$

c) if $x = y$, then

$$d(x, y) = 0.$$

The continuity of the symmetric d follows directly from its definition. We note that the topology τ_d (τ_d is the topology defined by the symmetric d in the sense of item 2) of Definition 1) induces on E the usual Euclidean topology, and on E^1 the discrete topology. The space X' is connected, locally connected, separable, locally satisfies the second axiom of countability, but does not satisfy the second axiom of countability. Consequently, it is not metrizable. Now let $X = X_1 \cup X_2$ be a subspace of X', where X_1 is the subset of E consisting of all points with rational coordinates, and $X_2 = E^1$. It is easy to see that the space X (and therefore also the space X') is not normal.

By Theorem 4, the spaces X and X' constructed in Example 3 have refining sequences of covers.

In the paper of Jones (9) the following assertion is proved:

Under the assumption that $2^{\aleph_1} > 2^{\aleph_0}$, every normal separable space with a refining sequence of covers is metrizable. Thus the problem arose: can the assumption $2^{\aleph_1} > 2^{\aleph_0}$ be dispensed with in the cited theorem of Jones? In connection with this we shall prove the following theorem:

Theorem 7. Every nonmetrizable, regular, separable space with a refining sequence of covers contains a nonmetrizable subspace Z, representable in the form $Z = Z_1 \cup Z_2$, where Z_1 is countable and Z_2 is discrete in Z. We note that Z is locally countable.

Theorem 8. If a space X is Δ-metrizable by a continuous Δ-metric, then X is metrizable.

Example 4. The arrow space is an example of a nonmetrizable hereditarily finally compact separable space, Δ-metrizable by a semicontinuous Δ-metric.

Lemma 6. If both an o-metric p defined on a set X and its conjugate q $(q(x,y) = p(y,x))$ are strong o-metrics, then every set U open in the topology τ_p (τ_q) is a set of type F_σ in the topology τ_q (τ_p).

Lemma 7. If p is a Δ-metric on a set X, and if the topologies τ_p and τ_q (where $q(x,y) = p(y,x)$ is the Δ-metric conjugate to p) are comparable by inclusion, then each of the spaces (X, τ_p) and (X, τ_q) has a refining sequence of covers, and at least one of them is metrizable.

Theorem 9. Every T_1-space X with a uniform base* is Δ-metrizable by a Δ-metric p satisfying the conditions: a) for any three points $x, y, z \in X$,

$$p(x,y) \leq \max\{p(x,z), p(z,y)\};$$

b) $\tau_p \subset \tau_q$, where q is the Δ-metric conjugate to p.

Therefore the space (X, τ_q) is metrizable and zero-dimensional. We note that $p = p(B)$ corresponds uniquely to any uniform base B of the space X.

Theorem 10. If X is a normal space with a uniform base, then X is o-metrizable by a lower semicontinuous o-metric ρ satisfying the following condition: for any three points $x, y, z \in X$ the inequality

$$\rho(x,y) \leq 2(\rho(x,z) + \rho(z,y))$$

holds.

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* A base $B = \{U_\alpha \mid \alpha \in A\}$ of a space X is called uniform if, for every point $x \in X$ and for every neighborhood O_x of it, the cardinality of the set

$$M_{x,U} = \{\alpha \in H \mid x \in U_\alpha,\ U_\alpha \setminus O_x \neq \Lambda\}$$

is finite (P. S. Aleksandrov ([10])).

Note: Figure translations are in progress. See original paper for figures.