# A MEAN-VALUE FORMULA FOR FUNDAMENTAL FUNCTIONS OF THE BELTRAMI OPERATOR AND AN EXACT ESTIMATE OF THE SUM OF SQUARES OF FUNDAMENTAL FUNCTIONS

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## Abstract

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# A MEAN-VALUE FORMULA FOR FUNDAMENTAL FUNCTIONS OF THE BELTRAMI OPERATOR AND AN EXACT ESTIMATE OF THE SUM OF SQUARES OF FUNDAMENTAL FUNCTIONS

*(Presented by Academician A. N. Tikhonov on 8 VII 1969)*

Let in an $N$-dimensional domain $G$ there be given the so-called Beltrami operator, i.e., a differential operator of the form

$$Bu = \frac{1}{\sqrt{g}} \sum_{i,k=1}^{N} \frac{\partial}{\partial x_k} \left( \sqrt{g}\, g^{ik} \frac{\partial u}{\partial x_i} \right). \tag{1}$$

We shall assume that the coefficients $g^{ik}$ of this operator are continuously differentiable in the domain $\Omega$, and that the quantity $1/g$ is a strictly positive and bounded determinant of the positive definite matrix $\|g^{ik}\|$.

As is well known, with each operator (1) there is associated a Riemannian metric with line element

$$ds^2 = \sum_{i,k=1}^{N} g_{ik}\, dx_i\, dx_k,$$

where the coefficients $g^{ik}$ of the Beltrami operator $B$ are equal to the algebraic complements of the elements $g_{ik}$ of the matrix $\|g_{ik}\|$, divided by the determinant of this matrix.

We introduce the concept of a fundamental system of functions of the Beltrami operator. Suppose that the domain $\Omega$ is a subdomain of some other $N$-dimensional domain $G$. Suppose further that the positive function $g(x)$, defined by the operator (1), has in some way been continued beyond the boundary of the domain $\Omega$ and is a function defined in the whole domain $G$.

**Definition.** A complete system $\{u_k(x)\}$, orthonormal in the domain $G$ with weight $\sqrt{g(x)}$, will be called a **fundamental system of functions** (f.s.f.) of the Beltrami operator $B$ in the subdomain $\Omega$, if each function $u_k(x)$ belongs in the open domain $\Omega$ to the class $C^{(2)}$ and, for some nonnegative number $\lambda_k$, satisfies inside $\Omega$ the equation $Bu_k + \lambda_k u_k = 0$.

The numbers $\lambda_k$ will henceforth be called **fundamental numbers**, and the corresponding functions $u_k(x)$ **fundamental functions**.

The concept of an f.s.f. of the Beltrami operator includes as a special case both the multiple trigonometric system and the system of eigenfunctions of any of the boundary-value problems both for the Beltrami operator and for the Laplace operator (corresponding to the "Euclidean" case). This concept also covers the system of eigenfunctions of any nonnegative self-adjoint extension of the minimal operator generated in $L_2$ by the operator $-B$.

At the same time, the study of an f.s.f. frees one from prescribing boundary conditions in any form and makes it possible to make no assumptions

both on the smoothness of the fundamental functions outside the domain $\Omega$, and on the smoothness of the boundary of the domain $G$.

If $L$ is a self-adjoint elliptic operator

$$Lu = \sum_{i,k=1}^{N} \frac{\partial}{\partial x_k}\left[a_{ik}(x)\frac{\partial u}{\partial x_i}\right], \tag{2}$$

then, for $N \neq 2$, the fundamental functions corresponding to the equation $Lu + \lambda\rho(x)u = 0$ with weight $\rho(x) = a^{1/(N-2)}$, where $1/a = \det\|a_{ik}\|$, are fundamental functions of the Beltrami operator (1) for $g^{ik} = a^{1/(2-N)}a_{ik}$, $g = 1/\det\|g^{ik}\|$. Thus, for $N \neq 2$, an arbitrary self-adjoint elliptic operator (2) is reduced to the Beltrami operator if the fundamental functions of this operator are taken with a definite weight $\rho(x)$ depending on its coefficients.

The first result of our work is the establishment, for the fundamental functions of the Beltrami operator in the subdomain $\Omega$, of a mean-value formula generalizing the well-known Weber formula for regular solutions of the equation $\Delta u + \lambda u = 0$ (see (1), p. 289). To establish the indicated formula, we fix an arbitrary point $x$ of the open domain $\Omega$ and, keeping in mind the Riemannian metric corresponding to the operator (1), fix a positive number $R$, smaller than the minimum of the geodesic distance from the point $x$ to the boundary of $\Omega$. In a neighborhood of the point $x$ we introduce the polar geodesic coordinates $r, \theta_1, \dots, \theta_{N-1}$ corresponding to the indicated Riemannian metric.

We note that the volume element in the indicated polar coordinate system can be written in the form $A(r)\Phi(\theta_1, \dots, \theta_{N-1})\, dr\, d\theta_1 \dots d\theta_{N-1}$, where $A(r)$ is the area of the surface of the Riemannian sphere of radius $r$ with center at the point $x$, and $\Phi(\theta_1, \dots, \theta_{N-1})$ is a nonnegative function only of the "angular" coordinates, moreover such that

$$\int \cdots \int \Phi(\theta_1, \dots, \theta_{N-1})\, d\theta_1 \dots d\theta_{N-1} = 1.$$

Let now $u_\lambda(x)$ be a fundamental function of the Beltrami operator in the subdomain $\Omega$, corresponding to the fundamental number $\lambda$. In other words, $u_\lambda(x)$ is, in the domain $\Omega$, a regular solution of the equation $Bu + \lambda u = 0$.

Denote by $\Phi_\lambda(x, r)$ the mean value of the fundamental function $u_\lambda(y)$ over all "angular" coordinates on the surface of the Riemannian sphere of radius $r$ with center at the point $x$, i.e. set

$$\Phi_\lambda(x, r) = \int \cdots \int u_\lambda(r, \theta_1, \dots, \theta_{N-1}) \Phi(\theta_1, \dots, \theta_{N-1})\, d\theta_1 \dots d\theta_{N-1}.$$

We prove that for any $r \le R$ the mean-value formula is valid

$$\Phi_\lambda(x, r) = u_\lambda(x) \bar{\varphi}_\lambda(r), \tag{3}$$

in which $\bar{\varphi}_\lambda(r)$ denotes that regular solution of the ordinary differential equation

$$\varphi''(r) + \frac{A'(r)}{A(r)} \varphi'(r) + \lambda \varphi(r) = 0, \tag{4}$$

which becomes equal to unity at $r = 0$*.

Of course, it should be noted that the area $A(r)$ of the Riemannian sphere of radius $r$ with center at the point $x$ (and, consequently, also the coefficient $A'(r)/A(r)$ of equation (4)) is, when the point $x$ varies, generally speaking a function not only of $r$, but also of $x$.

The question naturally arises of studying the behavior of the above-mentioned solution $\bar{\varphi}_\lambda(r)$ of equation (4).

* Along the way, uniqueness of such a solution is established.

The second result of our work is the establishment, for $\overline{\varphi}_\lambda(r)$, of an asymptotic formula uniform with respect to the collection $x, r$ (for large values of $\lambda$). To obtain the indicated formula we used a method close to that developed in the works ($^{2-4}$).

We have proved the following assertion.

**Theorem 1.** *Let the coefficients $g^{ik}$ of the Beltrami operator $B$ belong to the class $C^2(\Omega)$, and let $\Omega'$ be an arbitrary strictly interior subdomain of the domain $\Omega$, and let the positive number $R$ be smaller than the minimum of the geodesic* *distance between the boundaries of the domains $\Omega$ and $\Omega'$. Then for any point $x$ of the subdomain $\Omega'$ and any $r$ from the segment $0 \leq r \leq R$ the formula*

$$\overline{\varphi}_\lambda(r) = 2^{(N-1)/2}\sqrt{\Gamma\left(\frac{N}{2}\right)\pi^{N/2}}\sqrt{\frac{r}{A(r)}}\left[\lambda^{-(N-2)/4}J_{(N-2)/2}(r\sqrt{\lambda}) + \delta_\lambda(r)\right], \tag{5}$$

*holds, and for $\delta_\lambda(r)$, uniformly with respect to the collection $x, r$, for $x \in \Omega'$, $0 \leq r \leq R$, either of the following two estimates holds:*

$$|\delta_\lambda(r)| \leq Cr^{N/2}, \qquad |\delta_\lambda(r)| \leq \frac{C}{\lambda^{N/4}}\frac{\log^+ r\sqrt{\lambda}}{1+(r\sqrt{\lambda})^{1/2}}. \tag{6}$$

*(Here the symbol $\log^+ x$ has the following meaning: $\log^+ x = \max\{1, \log x\}$.)*

Formula (5) consists of two terms, the first of which corresponds to the Euclidean case (i.e., the case of the Laplace operator), while the second, by virtue of the estimates (6), has higher order in $1/\lambda$. It should also be noted that the first of the estimates (6) is effective for $r < 1/\sqrt{\lambda}$, and the second for $r > 1/\sqrt{\lambda}$.

Let us now consider a completely arbitrary fundamental system of functions of the Beltrami operator in the subdomain $\Omega$, not excluding the existence of finite points of condensation in the set of fundamental numbers $\{\lambda_k\}$.

With the aid of the mean-value formula (3) and Theorem 1 we have proved the following assertion.

**Theorem 2.** *For any $\mu > 0$, uniformly with respect to $x$ in each strictly interior subdomain $\Omega'$ of the domain $\Omega$, the estimate*

$$\sum_{|\sqrt{\lambda_k}-\mu|\leq 1} u_k^2(x) = O(\mu^{N-1}) \tag{7}$$

*holds.*

*If the geodesic distance from the point $x$ to the boundary of the domain $\Omega$ is denoted by $\rho(x)$, then the estimate (7) can be sharpened as follows:*

$$\sum_{|\sqrt{\lambda_k}-\mu|\leq 1} u_k^2(x) \leq \frac{C}{\rho(x)}\mu^{N-1}, \qquad C = \text{const}.$$

Let us note that the estimate (7) (in order with respect to $\mu$) is exact. The order indicated on the right-hand side of (7) is attained even in the case when on the left-hand side of (7) there stands only one term—one eigenfunction of the first or the second boundary-value problem for the Laplace operator in an

$N$-dimensional ball, possessing radial symmetry and taken at the center of this ball.

In the case when, on the segment $(\mu-1)^2 \leq \lambda_k \leq (\mu+1)^2$, there lie points of condensation of the set $\{\lambda_k\}$, the left-hand side of (7) is a convergent series.

**Corollary of Theorem 2.** *For any* $\delta > 0$ *the series*

$$\sum_{\lambda_k \geq 1} \frac{u_k^2(x)}{\lambda_k^{N/2+\delta}} \tag{8}$$

*possesses a uniformly bounded family of partial sums in every strictly interior subdomain* $\Omega'$*.*

If the fundamental numbers have no finite accumulation points, series (8) converges uniformly in every strictly interior subdomain $\Omega'$. To this one should add that, for $\delta = 0$, series (8) diverges at every interior point of the domain $\Omega$.

**Remark.** All the results of the present paper remain valid if, in the definition of f.f., instead of the domain $G$ and its subdomain $\Omega$, one takes a closed $N$-dimensional Riemannian manifold $G$ and a compact $N$-dimensional Riemannian manifold $\Omega$ embedded in it, on which the Laplace–Beltrami operator is defined. In this case formula (5), estimate (7), and the convergence of series (8) will be uniform on any Riemannian submanifold $\Omega'$ embedded in $\Omega$, the distance of all points of which from the boundary of $\Omega$ exceeds a positive constant (and if $G = \Omega$ is a compact manifold without boundary, then uniform on the whole manifold $\Omega$).

In conclusion, we note that a number of works, beginning with the rather early work of B. Feller ($^5$) and ending with the recent work of B. Fussaro ($^6$), have been devoted to the study of mean-value formulas for regular solutions of elliptic equations by introducing Riemannian metrics, but these works pursued quite different aims.

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## REFERENCES

[1] R. Courant, *Partial Differential Equations*, Moscow, 1964.
[2] F. Tricomi, *Differential Equations*, IL, 1962.
[3] A. A. Dorodnitsyn, UMN, 7, No. 6, 3 (1952).
[4] D. P. Kostomarov, *Matem. sbornik*, 45, No. 1, 17 (1958).

[5] W. Feller, UMN, 8, 232 (1941).
[6] B. A. Fussaro, J. Math. and Mech., 18, No. 7, 603 (1969).

*Note: Figure translations are in progress. See original paper for figures.*

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