SOME NONSTATIONARY PROBLEMS IN THE THEORY OF FINITE-AMPLITUDE WAVES IN DISSIPATIVE MEDIA

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Abstract

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SOME NONSTATIONARY PROBLEMS IN THE THEORY OF FINITE-AMPLITUDE WAVES IN DISSIPATIVE MEDIA

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Numerous works in acoustics, radiophysics, and magnetohydrodynamics have been devoted to nonlinear wave processes in nondispersive dissipative media. The propagation of finite-amplitude waves is, as a rule, studied on the basis of approximate equations, and the approximate equations themselves admit exact analytical solutions only in exceptional cases. In this connection, for example, in acoustics, alongside the principal small parameter—the Mach number—another parameter is considered, the Reynolds number (Re). For media in which nonlinear effects predominate ($\mathrm{Re} \gg 1$), and for media in which dissipative effects predominate ($\mathrm{Re} \ll 1$), different methods are used for solving the approximate equations ([1]). In the practically most interesting case of values $\mathrm{Re} \gg 1$, an exact solution over the whole region of wave propagation has been obtained only for plane waves ([2]). For these same waves, two nonstationary problems have been solved: the propagation of a single disturbance prescribed at the system input in the form $v = v_0 \,\mathrm{th}(\tau/\tau_0)$, where v is the hydrodynamic velocity, and the propagation of a single impulse of triangular profile ([3]). Spatially symmetric flows were studied on the basis of "matching" solutions pertaining to systems with $\mathrm{Re} \to \infty$ and "quasistationary" solutions ([4]). This method of stepwise simplification of the equations made it possible to study the propagation of diverging and converging cylindrical and spherical stationary waves. Nonstationary problems in dissipative spatially symmetric systems have not been considered at all.

In the present work, for important particular cases, nonlinear partial differential equations describing plane and cylindrically symmetric waves are reduced to ordinary differential equations. Exact solutions are obtained which, in the case of plane waves, make it possible to consider the propagation of single pulses of various profiles. For cylindrical waves the solution is obtained in parametric form and, as an analytical consideration of limiting values of the parameter shows, is an analogue of a quasistationary solution. In contrast to the latter, the solution found in the present work is exact.

The approximate equations describing the propagation of finite-amplitude waves in liquids and gases in the plane and cylindrically symmetric cases have, respectively, the form

$$\partial v/\partial z - \alpha v\,\partial v/\partial \tau = \delta\,\partial^3 v/\partial \tau^2; \tag{1}$$

$$\partial w/\partial x - \alpha w\,\partial w/\partial \tau = (\delta x/2R_0)\,\partial^2 w/\partial \tau^2. \tag{2}$$

The notation in equations (1) and (2) corresponds exactly to the notation adopted in works ([2,4]). Let us note that the equations describing magnetoacoustic waves differ from equations (1) and (2) only in the values of the constant coefficients α and δ ([3,5]).

Introducing the substitution $v = \frac{1}{\sqrt{z_0+z}}\varphi\left(\frac{\tau_0+\tau}{\sqrt{z_0+z}}\right)$, we transform equation (1) to the form

$$\delta\,d\varphi/d\xi + \alpha\varphi^2/2 + \xi\varphi/2 = C_0, \tag{3}$$

where $\xi = (\tau_0+\tau)/\sqrt{z_0+z}$; C_0 is an integration constant which, from the condition that the function φ vanish at infinity, may be taken equal to zero. Integrating equation (3) and returning to the hydrodynamic velocity v, we find:

$$v = \frac{2}{\sqrt{\pi}}\,\frac{\sqrt{\delta}}{\alpha\sqrt{z_0+z}}\exp\left[-\frac{(\tau_0+\tau)^2}{4\delta(z_0+z)}\right]\left[\Phi\left(\frac{\tau_0+\tau}{\sqrt{4\delta(z_0+z)}}\right)+C\right]^{-1}. \tag{4}$$

The arbitrary constant C must be chosen so that the denominator in formula (4) does not vanish. The solution (4) is free of restrictions on the magnitude of the Reynolds number within the orders of smallness adopted in deriving equation (1). The pulse (4), prescribed at the input of the system ($z = 0$), propagates without distortion of form under the corresponding change in the scale of the accompanying coordinate, decreasing only in "amplitude." The solution found is essentially a self-similar solution for a dissipative system, if instead of the function v one introduces $v' = v\sqrt{z_0+z}$.

Fig. 1
Fig. 1

Fig. 2
Fig. 2

In Fig. 1, pulses are constructed on the basis of solution (4) for various values of the constant C, which characterizes the relation between the nonlinear and dissipative properties of the medium. The asymmetric pulse with a steep leading front corresponds to values $\mathrm{Re} \gg 1$; the gently sloping pulse corresponds to values $\mathrm{Re} \ll 1$, or to a pulse of any profile at sufficiently large distances from the input of the system. In this limiting case, the asymptotics of formula (4) coincides with expression (32) of work (3). The intermediate position in Fig. 1 is occupied by a pulse corresponding to values $\mathrm{Re} \sim 1$.

Equation (2), by means of the substitution $w = \varphi(\xi)$, where $\xi = \tau/x$, is reduced to the ordinary differential equation

$$\frac{\delta}{2R_0}\frac{d^2\varphi}{d\xi^2} + \alpha\varphi\frac{d\varphi}{d\xi} + \xi\frac{d\varphi}{d\xi} = 0. \tag{5}$$

Solving equation (5) and returning to the variable w, we obtain

$$w = -\frac{1}{\alpha}\left[\xi - \sqrt{\frac{2\delta}{R_0}}\sqrt{\eta(\xi) - Ce^{-2\eta(\xi)}}\right]; \tag{6}$$

$$\frac{d\eta}{d\xi} = \sqrt{\frac{2R_0}{\delta}}\sqrt{\eta(\xi) - Ce^{-2\eta(\xi)}}. \tag{7}$$

A more convenient form for analysis is the parametric form:

$$w = -\frac{c}{\alpha}\sqrt{\frac{2\delta}{R_0}}\int_{\eta_0}^{\eta}\frac{e^{-2\eta}}{\sqrt{\eta - Ce^{-2\eta}}}\,d\eta; \tag{8}$$

$$\frac{\tau}{x} = \sqrt{\frac{\delta}{2R_0}}\int_{\eta_0}^{\eta}\frac{d\eta}{\sqrt{\eta - Ce^{-2\eta}}}. \tag{9}$$

In Fig. 2 the subradical function $f(\eta) = \eta - Ce^{-2\eta}$ and its asymptote for $C = 0$ are plotted. Note that for the value of the constant $C = 0$, formulas (8), (9) give the solution of the linear equation $(\partial w/\partial x) - (x\delta/2R_0) \times (\partial^2 w/\partial\tau^2) = 0$, valid for systems characterized by an infinitely small Reynolds number. By assigning a definite value to the constant C, one may consider systems with an arbitrary relation between the nonlinear and dissipative properties of the medium, i.e., for arbitrary values of the Reynolds number, except infinitely large ones, when in the initial equation (2) one may set $\delta = 0$. The value of the parameter η_0 corresponds to the vanishing of the function $f(\eta)$. It is precisely in the neighborhood of the point η_0 that the nonlinear properties are manifested most strongly. As the Reynolds number decreases, the point $\eta \to 0$, and the function $f(\eta)$ deviates only weakly from a linear law. Therefore, of greatest interest is the investigation of the behavior of the solutions found, (8), (9), in the neighborhood of the point η_0, where the subradical function $f(\eta)$, with the aid of an expansion in the small parameter $\varepsilon = \eta - \eta_0$, can be represented in the form

$$f(\eta) = \eta - Ce^{-2\eta} = \varepsilon(1 + 2\eta_0). \tag{10}$$

Fig. 3

After replacing, by means of (10), the subradical expression in formulas (8), (9), integrating, and eliminating the parameter ε, we find

$$w = -\frac{\sqrt{\pi\delta}}{\alpha\sqrt{R_0}} \frac{\eta_0}{\sqrt{1+2\eta_0}} \Phi\left[\frac{\sqrt{1+2\eta_0}}{\sqrt{\delta/R_0}} \frac{\tau}{x}\right]. \tag{11}$$

The self-similar solutions (8), (9) are presented in Fig. 3 for different values of the Reynolds number. Curve 1 corresponds to the values Re $\gg$ 1, 2 to Re $\sim$ 1, and 3 to Re $\ll$ 1. The dependence on the value of the Reynolds number is taken into account implicitly by means of the parameter η_0, the root of the transcendental equation $\eta - Ce^{-2\eta} = 0$. Just as expressions (8), (9) gave the solution of the linear equation for $C = 0$, formula (11), when the parameter $\eta_0 \to 0$, also passes into the solution of the linear equation $(\partial w/\partial x) - (x\delta/2R_0)(\partial^2 w/\partial\tau^2) = 0$.

The solutions (8), (9) and the expression (11), valid for small values of the parameter ε, are equally applicable both to diverging and to converging cylindrically symmetric disturbances in the medium. Therefore, in contrast to solution (4), which reveals only a single tendency toward the "spreading" of pulses, here both a decrease (x increases) and an increase (x decreases) of the steepness of the front are possible. The latter occurs in signal focusing and is valid within the limits of the restrictions adopted in deriving the approximate equation (2) ([4]).

The exact solutions (8), (9) and the approximate solution (11), by virtue of their structure, are also of substantial interest when considering the propagation of sinusoidal disturbances in cylindrically symmetric systems. Defining the structure of the front of a weak shock wave, they can be used instead of the "quasi-stationary" solution introduced from physical considerations without sufficient mathematical justification ([4,5]).

Thus, on the basis of reducing nonlinear second-order partial differential equations to ordinary differential equations, self-similar solutions have been obtained that describe the propagation of a single pulse in the case of plane waves and a single "discontinuity" in the case of cylindrically symmetric waves. The exact solutions found are free of restrictions on the magnitude of the Reynolds number.

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Note: Figure translations are in progress. See original paper for figures.