DECAY OF SOLUTIONS OF THE SECOND EXTERIOR BOUNDARY-VALUE PROBLEM FOR THE WAVE EQUATION WITH TWO SPACE VARIABLES

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Abstract

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L. A. MURAVEI

DECAY OF SOLUTIONS OF THE SECOND EXTERIOR BOUNDARY-VALUE PROBLEM FOR THE WAVE EQUATION WITH TWO SPACE VARIABLES

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Let the function $u(x,t)$ in the exterior Ω of a convex contour Γ satisfy the conditions

$$\partial^2 u(x,t)/\partial t^2 - \Delta u(x,t) = 0, \qquad x \in \Omega, \quad t > 0; \tag{1}$$

$$u\big|_{t=0} = f_1(x), \qquad \partial u/\partial t\big|_{t=0} = f_0(x), \qquad x \in \overline{\Omega} \tag{2}$$

and, on the contour Γ, the condition

$$\partial u/\partial n = 0, \qquad x \in \Gamma, \quad t \geqslant 0, \tag{3}$$

where $x = (x_1, x_2)$, $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$, and n is the unit normal to the contour Γ, exterior with respect to Ω.

This note is devoted to the study of the asymptotic behavior of the solution $u(x,t)$ of problem (1)–(3) as $t \to \infty$.

Problem (1)–(3) for the case of three space variables was studied by V. P. Mikhailov in [1], where it was shown that if the functions f_0, f_1 are finite and the surface Γ has positive Gaussian curvature, then the solution $u(x,t)$ decays on every compact set in x as $e^{-\alpha t}$. In the case of three space variables, for the first boundary-value problem ($u|_\Gamma = 0$), K. Morawetz [2] proved that if the functions f_0, f_1 are finite and the surface Γ is star-shaped, the solution $u(x,t)$ decays on every compact set in x as $1/t$. Using this result, P. Lax, K. Morawetz, and R. Phillips [3] established, under the assumptions of [2], the exponential decay of the solution on every compact set in x as $t \to \infty$. A detailed survey of results in this direction is given in the paper of T. I. Zelenyak and V. P. Mikhailov [4].

We shall consider the solution $u(x,t)$ of problem (1)–(3) under the following assumptions: the contour $\Gamma \in C^3$, and the radius of curvature $R(s)$ (s is arc length of the curve) satisfies the inequalities

$$0 < R_0 \leqslant R(s) \leqslant R_1 < \infty. \tag{4}$$

The functions $f_\alpha(x) \in C^{2+\alpha}(\overline{\Omega})$, $\alpha = 0, 1$, and satisfy the conditions

$$\left|\partial^{\alpha_1+\alpha_2} f_\alpha(x)/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\right| \leqslant C_\alpha e^{-B_\alpha |x|}, \qquad x \in \overline{\Omega}; \tag{5}$$

$$\partial f^\alpha/\partial n = 0, \qquad x \in \Gamma, \tag{6}$$

where B_α, C_α, $\alpha = 0, 1$, are certain positive constants. In what follows we shall assume that the functions $f_\alpha(x)$, $\alpha = 0, 1$, have been extended to the entire plane E_2 with preservation of smoothness and of estimate (5).

Represent the function $u(x,t)$ in the form

$$u(x,t) = u_0(x,t) + u_1(x,t), \tag{7}$$

where $u_\alpha(x,t)$, $\alpha = 0, 1$, are the solutions of problem (1)–(3) for $f_{1-\alpha} \equiv 0$.

Theorem. If conditions (4)–(6) are fulfilled, then for every $x \in \overline{\Omega}$ there exist positive constants $T = T(x, \Gamma, f_\alpha) < \infty$ and $C = C(x, \Gamma, f_\alpha) < \infty$ such that, for $t \geqslant T$, for the functions $u_\alpha(x,t)$, $\alpha = 0, 1$,

from (7) the estimate holds

$$|u_\alpha(x,t)| \leq C/t^{1+\alpha}, \qquad \alpha = 0, 1. \tag{8}$$

If the supports of the functions f_α, $\alpha = 0, 1$, are contained in a compact set $Q \subset \overline{\Omega}$, the functions $f_\alpha \in C^{2+\alpha}(Q)$ and satisfy condition (6), then for any $x \in \overline{\Omega}$ there is a constant $0 < D = D(x, \Gamma, f_\alpha, Q) < \infty$ such that, as $t \to \infty$, the equality

$$u_\alpha(x,t) = \frac{(-1)^\alpha}{2\pi t^{1+\alpha}} \iint_Q f_\alpha(y)\, dy_1 dy_2 + \tilde{u}_\alpha(x,t), \tag{9}$$

holds, where

$$|\tilde{u}_\alpha(x,t)| \leq D \ln t / t^{2+\alpha}, \qquad \alpha = 0, 1.$$

Consider in E_2 the functions $w_\alpha(x,t),\ \alpha=0,1$, the solutions of the Cauchy problem for equation (1) with the functions $f_\alpha,\ \alpha=0,1$, as the initial data; as is known,

$$w_\alpha(x,t)=\frac{\partial^\alpha}{\partial t^\alpha}\left\{\frac{1}{2\pi}\int_{|y|\le t}\frac{f_\alpha(x+y)}{\sqrt{t^2-|y|^2}}\,dy_1dy_2\right\},\qquad \alpha=0,1,\quad x\in E_2,\quad t\ge 0.$$

Lemma 1. For any $x\in E_2$ there is a positive constant $C'_\alpha=C'_\alpha(x,f_\alpha)$ such that, as $t\to\infty$, for the functions $w_\alpha(x,t),\ \alpha=0,1$, the representations

$$w_\alpha(x,t)=\frac{(-1)^\alpha}{2\pi t^{1+\alpha}}\iint_{E_2}f_\alpha(y)\,dy_1dy_2+\tilde{w}_\alpha(x,t), \tag{10}$$

are valid, where

$$|\tilde{w}_\alpha(x,t)|\le C'_\alpha/t^{3+\alpha},\qquad \alpha=0,1.$$

We shall seek the functions $u_\alpha(x,t)$ in the form

$$u_\alpha(x,t)=w_\alpha(x,t)+v_\alpha(x,t),\qquad \alpha=0,1. \tag{11}$$

Then, for $x\in\Omega$, the functions $v_\alpha(x,t),\ \alpha=0,1$, satisfy equation (1), the zero initial conditions (2), and, for $x\in\Gamma$, the condition $\partial v_\alpha/\partial n=-\partial w_\alpha/\partial n,\ t\ge 0$.

Denote by $V_\alpha(x,k)$ the formal Laplace transform of the function $v_\alpha(x,t),\ \alpha=0,1$. The function $V_\alpha(x,k)$ is a solution of the problem

$$\Delta V_\alpha(x,k)+k^2V_\alpha(x,k)=0,\qquad x\in\Omega; \tag{12}$$

$$\partial V_\alpha/\partial n=F_\alpha(x,k),\qquad x\in\Gamma; \tag{13}$$

$$\partial V_\alpha/\partial|x|-ikV_\alpha=e^{ik|x|}o(|x|^{-1/2})\qquad \text{as } |x|\to\infty, \tag{14}$$

where $\alpha=0,1,\ \operatorname{Im}k>0$,

$$F_\alpha(x;k)=-\int_0^\infty e^{ikt}\,\frac{\partial w_\alpha(x,t)}{\partial n}\bigg|_{x\in\Gamma}\,dt.$$

Starting from the form of the functions $F_\alpha,\ w_\alpha$ and properties (5)–(6) of the functions f_α, one proves

Lemma 2. The functions $F_\alpha(x,k)$, $\alpha=0,1$, are continuous in $x\in\Gamma$ and analytic in k in the domain

$$K_b \equiv \{k:\ -b \le \operatorname{Im} k \le b, \quad \operatorname{Re} k \ne 0, \quad \operatorname{Im} k \le 0\},$$

where $0<b<\min(B_0,B_1)$ (B_0, B_1 are the constants from (5)); moreover, for $k\in K_b\cap\{|k|\le b\}$ the representations

$$F_\alpha(x,k) = \frac{(-1)^{1+\alpha}(ik)^{1+\alpha}}{4}\iint_{E_2'} f_\alpha(x+y)\frac{(n,y)}{|y|}H_1^{(1)}(k|y|)\,dy_1dy_2, \qquad \alpha=0,1,$$

and for $k\in K_b\cap\{|k|\ge b\}$ the estimates

$$|F_\alpha(x,k)| \le \frac{C''_\alpha}{|k|\,|\operatorname{Re} k|^{1/2}}, \qquad \alpha=0,1,$$

hold, where $C''_\alpha = C''_\alpha(\Gamma, f_\alpha, b)$ are certain positive constants.

The solution of problem (12)–(14) can be represented in the form

$$V_\alpha(x,k) = \int_\Gamma F_\alpha(y,k)\,G(x,y,k)\,dy, \tag{15}$$

where $G(x,y,k)$ is the Green' s function of problem (12)–(14) (its existence for $\operatorname{Im} k\ge 0$ follows from [5]).

Under assumptions (4) on the contour Γ, the following basic lemma 3 holds.

Lemma 3. For any $y\in\Gamma$ and fixed $x\in\overline{\Omega}$, the function $G(x,y,k)$ is analytic in the domain $K_{b'}$, for some $0<b'=b'(\Gamma)$. Moreover, in the domain $K_{b'}\cap\{|k|\ge b'\}$ the estimate

$$|G(x,y,k)| \le \frac{C\,|k|^{-1/6}}{|x-y|^{1/2}}\,(1+e^{-D\operatorname{Im} k})e^{-B|x-y|\operatorname{Im} k}, \tag{16}$$

holds, and in the domain $K_{b'}\cap\{|k|\le b'\}$ the representation

$$G(x,y,k) = G^{(0)}(x,y) - \frac{1}{2\pi}\ln(\gamma k) + G^{(1)}(x,y,k),$$

holds, where $G^{(0)}(x,y)$ is the Green' s function of the corresponding problem for the Laplace equation; $G^{(1)}(x,y,k)$ is a function analytic in the domain $K_{b'}\cap\{|k|\le b'\}$, for which the estimate

$$|G^{(1)}(x,y,k)| \le C^{(1)}|k \ln k|$$

holds; the constants $B, C, D > 0$ depend on x, Γ, b'; $C^{(1)} = C^{(1)}(x, \Gamma)$ is a certain positive constant, and γ is a certain complex number.

The main difficulty in proving the lemma is establishing estimate (16) as $k \to \infty$. For this purpose an analytic continuation of the Green's function of problem (12) –(14) is constructed for a circle of curvature to the contour Γ at an arbitrary point of it. For sufficiently large $|\operatorname{Re} k|$, this continuation can be obtained in a certain strip containing the axis $\operatorname{Im} k = 0$. Using the properties of this continuation, for the function $G(x,y,k)$ in this strip one can obtain an integral equation of the type introduced in [6] in the study of the short-wave diffraction problem. For the kernel of the integral equation an estimate is established that makes it possible to solve it by the method of iterations and to establish inequality (16).

From lemmas 2, 3 it follows that the functions $V_\alpha(x,k)$, $\alpha = 0,1$, in (15) are analytic in the domain K_β, $\beta = \min\{b, b'\}$, and for $k \in K_\beta \cap \{|k| \ge \beta\}$ the estimate

$$|V_\alpha(x,k)| \le \frac{C}{|k|^{7/6}|\operatorname{Re} k|^{1/2}}(1 + e^{-D \operatorname{Im} k})e^{-B|x| \operatorname{Im} k}, \qquad \alpha = 0,1, \tag{17}$$

holds for them; the constants $B, C, D > 0$ depend on x, Γ, f_α.

The functions $v_\alpha(x,t)$ from (11) are represented in the form

$$v_\alpha(x,t) = \frac{1}{2\pi}\int_{\operatorname{Im} k = \varepsilon} e^{-ikt} V_\alpha(x,k)\, dk, \qquad \alpha = 0,1,$$

where ε is an arbitrary number from the interval $(0, \beta)$.

On the basis of estimate (17), for any $t > 0$ the integral over the contour $\operatorname{Im} k = \varepsilon$ can be transformed to the form

$$v_a(x,t) = \frac{i}{2\pi}\int_0^\beta e^{-\eta t}[V_a(x, \eta e^{-i\pi/2}) - V_a(x, \eta e^{3\pi/2})]\, d\eta +$$

$$+\frac{e^{-\beta t}}{2\pi}\int_0^\infty e^{it\xi}[V_a(x, -\beta + i\xi) - V_a(x, -\beta - i\xi)]\, d\xi = v_a^{(0)}(x,t) + v_a^{(1)}(x,t)$$

$$(a = 0,1).$$

For $t \ge T = 2(D + B|x|)$ (B, D are the constants from (17)) the estimate

$$|v_a^{(1)}(x,t)| \le C^{(1)}(x, \Gamma, f_a)e^{-\beta t/2}. \tag{18}$$

Using the form of the function $G(x, y, k)$ and of the functions $F_a(x, k)$, $a = 0, 1$, for $k \in K_\beta \cap \{|k| \le \beta\}$, as well as the restrictions (5)–(6) on the functions f_a, $a = 0, 1$, we arrive, as $t \to \infty$, at the estimate

$$|v_a^{(0)}(x,t)| \le C(x, \Gamma, f_a)/t^{1+a}, \qquad a = 0, 1. \tag{19}$$

If the functions f_a, $a = 0, 1$, have compact support Q, then as $t \to \infty$ one can obtain the expansion

$$v_a^{(0)}(x,t) = \frac{(-1)^{1+a}}{2\pi t^{1+a}} \iint_{E_2 \setminus Q} f_a(y)\, dy_1 dy_2 + \tilde{v}_a(x,t), \tag{20}$$

where

$$|\tilde{v}_a(x,t)| \le \widetilde{C}(x, \Gamma, f_a, Q) \ln t / t^{2+a}, \qquad a = 1, 0. \tag{21}$$

We shall now establish the properties, formulated in the theorem, of the solution $u(x, t)$ of problem (1)–(3). Estimate (8) follows from representation (11) for the functions u_a and from the properties (10), (18), (19) of the functions w_a, v_a, $a = 0, 1$. Equality (9) follows from the form (11) of the functions u_a and from the properties (10), (18), (20)–(21) of the functions w_a, v_a, $a = 0, 1$, valid as $t \to \infty$.

Moscow Institute of Physics and Technology

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Note: Figure translations are in progress. See original paper for figures.