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Abstract

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INVESTIGATION OF SOME SINGULAR INTEGRAL OPERATORS AND THE CORRESPONDING NONLINEAR EQUATIONS

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In the present note we give some inequalities for the operators

$$A_1 f \equiv \int_\Gamma \frac{f(t_0,\tau)}{\tau - t}\, d\tau \equiv A(t_0, t), \qquad (a_1)$$

$$A_2 f \equiv \int_\Gamma \frac{f(t,\tau)}{\tau - t}\, d\tau \equiv A(t, t) \qquad (a_2)$$

and theorems on the existence and uniqueness of solutions of the equations

$$u(t_0, t) = \lambda K_1 u \equiv \lambda \int_\Gamma \frac{K[t_0, \tau, u(t_0,\tau)]}{\tau - t}\, d\tau, \qquad (a_1^0)$$

$$u(t) = \lambda K_2 u \equiv \lambda \int_\Gamma \frac{K[t, \tau, u(\tau)]}{\tau - t}\, d\tau \qquad (a_2^0)$$

in certain subsets of the space L_p.

I. Let Γ be a closed or open rectifiable Jordan curve l such that, for any $u(t) \in L_p(\Gamma)$, the generalized Riesz inequality holds,

$$\|S_\varphi\| \leq A_p \|\varphi\|_{L_p}, \qquad \text{where} \qquad S_\varphi \equiv \int_\Gamma \frac{\varphi(\tau)}{\tau - t}\, d\tau. \qquad \text{(b)}$$

Examples of contours for which (b) holds are given in works ([1–3]). Let $u(t_0, t) \in L_p(\Gamma \times \Gamma)$; ξ and ζ are arc abscissas of the line Γ. Denote

$$\|u\|_{L_p} \equiv \left\{ \int_\Gamma \int_\Gamma |u(t_0, t)|^p \, |dt_0| \, |dt| \right\}^{1/p},$$

$$w(u;\sigma_1,\sigma_2)=\sup_{\substack{0<h_1<\sigma_1\\0<h_2<\sigma_2}}\left\{\int_0^{l-\sigma_2}\int_0^{l-\sigma_1}|u[t(\xi+h),\tau(\zeta+h_2)]-u[t(\xi),\tau(\zeta)]|^p\,d\xi\,d\zeta\right\}^{1/p}.$$

If $f(t_0,t)\in L_p(\Gamma\times\Gamma)$, then from (b) it follows that

$$\|A_1 f\|_{L_p}\leq A_n\|f\|_{L_p\Gamma\times\Gamma}. \tag{1}$$

If one uses (b) and the generalized Minkowski inequality, an analogue of the Zygmund–Magaradze inequality is obtained:

$$w(A_1 f;\sigma_1,\sigma_2)\leq M_1\int_0^{\sigma_2}\frac{w(f,0,\xi)}{\xi}\,d\xi+M_2\sigma_2\int_{\sigma_2}^{l}\frac{w(f,0,\xi)}{\xi^2}\,d\xi+A_p w(f,\sigma_1,0). \tag{2}$$

Definition 1. A function $u(t_0,t)\in H^p_{\delta_1,\delta_2}(N,N_1,N_2)$ if

$$\|u\|_{L_p}\leq N, \tag{b_1}$$

$$w(u;\sigma_1,\sigma_2)\leq N_1\sigma_1^{\delta_1}+N_2\sigma_2^{\delta_2}. \tag{b_2}$$

It is evident that $H^p_{\delta_1,\delta_2}(N,N_1,N_2)$ is a compact set in L_p. Let

the function $K(t_0,t,u)$ ($t_0,t\in\Gamma\times\Gamma$, and u is an arbitrary point of the complex plane) satisfies the conditions:

$$\left\{\int_0^{l-\sigma_1}\int_0^{l-\sigma_2}|K\left[t(\xi+\sigma_1),t(\eta+\sigma_2),u(t(\xi),t(\eta))\right]\right.$$
$$\left.-K\left[t(\xi),t(\eta),u(t(\xi),t(\eta))\right]|^p\,d\xi\,d\eta\right\}^{1/p}\leq M_3\sigma_1^{\delta_1}+M_4\sigma_2^{\delta_2}, \tag{3}$$

$$\left\{\iint_L\iint_L|K[t_0,t,u(t_0,t)]-K[t_0,t,v(t_0,t)]|^p\,|dt|\,|dt_0|\right\}^{1/p}\leq$$
$$\leq M_5\left\{\iint_L\iint_L|u(t_0,t)-v(t_0,t)|^p\,|dt_0|\,|dt|\right\}^{1/p} \tag{4}$$

for arbitrary $u(t_0,t),v(t_0,t)$ from $L_p(\Gamma\times\Gamma)$. Condition (4) is fulfilled, in particular, if for almost all t_0,t from Γ one has

$$|K(t_0,t,u)-K(t_0,t,v)| \leq D(t_0,t)|u-v|, \tag{4'}$$

where u, v are arbitrary complex numbers, and $D(t_0,t) \in L_q(\Gamma\times\Gamma)$ $(1/p+1/q=1)$.

From (1) and (2) it follows

Theorem 1. If $K(t_0,t,u)$ satisfies conditions (3), (4), then the operator $\lambda K_1 u$ acts in $H^p_{\delta_1,\delta_2}(N,N_1,N_2)$ and is a contraction operator in the metric of the space $L_p(\Gamma,\Gamma)$ for all $|\lambda| < \mu_1$, where μ_1 is sufficiently small depending on $N, N_1, N_2, (M_1-M_5), A_p$.

From this theorem, in turn, it follows

Theorem 2. If $|\lambda| < M_2$, then equation (a_1^0) has a solution in $H^p_{\delta_1,\delta_2}(N,N_1,N_2)$, unique in all of $L_p(\Gamma\times\Gamma)$, and this solution can be found by the method of Picard successive approximations, if the initial function $u(t_0,t)$ is taken from $H^p_{\delta_1,\delta_2}(N,N_1,N_2)$.

Remark 1. If the functions $f(t_0,t)$, $K(t_0,t,u)$, $u(t_0,t)$ are defined with respect to t_0 not on Γ, but on an arbitrary rectifiable curve Γ_1, situated arbitrarily relative to Γ, then inequalities (1)–(4), and consequently both theorems, remain valid.

Remark 2. If Γ is an arbitrary rectifiable Jordan curve satisfying only the condition

$$\sup_{t_1,t_2\in M} \frac{S(t_1,t_2)}{|t_1-t_2|} = M < \infty, \tag{c}$$

and $f(t_0,t) \in H^p_{\delta_1,\delta_2}(N,N_1,N_2)$, then, using the inequalities of Hölder and Minkowski, we obtain

$$\|A_1 f\|_{L_p(\Gamma\times\Gamma)} \leq M\cdot M_1 \|f\|_{L_p} \ln\left(\|f\|_{L_p}\right) + \frac{2M}{\delta_2}\|f\|_{L_p}, \tag{5}$$

$$w(A_1 f;\sigma_1,\sigma_2) \leq M_1 \int_0^{\sigma_2} \frac{w(f,0,\xi)}{\xi}\,d\xi + M_2\sigma_2 \int_{\sigma_2}^{l} \frac{w(f,0,\xi)}{\xi^2}\,d\xi +$$

$$+M_7 w(f,\sigma_1,0)\ln\frac{1+\sigma_1}{\sigma_1}. \tag{6}$$

But from these inequalities it does not follow that the operator $\lambda A_1 f$ acts in $H^p_{\delta_1,\delta_2}(N,N_1,N_2)$ for any $\lambda \neq 0$.

II. For the operator (a_2), generally speaking, on no contour is there an analogue of inequality (4). This assertion can be confirmed by examples, which we shall not give here.

Let the contour Γ satisfy (c). Denote

$$\|\bar{f}\|_{L_p} = \left\{ \int_L |f(t,t)|^p \, |dt| \right\}^{1/p}; \qquad \|\tilde{f}\|_{L_p} = \sup_{0 \le \xi \le l} \left\{ \int_0^l |f(t(s), t(s+\xi))|^p \, ds \right\}^{1/p},$$

$$\overline{w}(f; \sigma_1, \sigma_2) = \sup_{\substack{0 < h_1 \leqslant \sigma_1 \\ 0 < h_2 \leqslant \sigma_2}} \left\{ \int_0^l |f(t(s+h_1), t(s+h_2)) - f(t(s), t(s))|^p \, ds \right\}^{1/p},$$

$$\widetilde{w}(t, \sigma_1) = \sup_{\substack{0 < h_1 \leqslant \sigma \\ 0 < \xi \leqslant l}} \left\{ \int_0^l |f(t(s+h_1), t(s+\xi)) - f(t(s), t(s+\xi))|^p \, ds \right\}^{1/p}.$$

Here we shall regard the function $f[t_0(s), t(s)]$ as equal to zero outside the square $[0,l] \times [0,l]$.

If $\|\tilde{f}\|_{L_p} < \infty$ and $\overline{w}(f, \sigma, \sigma) \leqslant \overline{M} \sigma^\alpha$, then the inequalities

$$\|A_2\|_{L_p} \leqslant \frac{M \cdot \overline{M} \sigma}{\alpha} \|\tilde{f}\|_{L_p} + \Omega_1(f), \tag{7}$$

$$w(A_2 f, \sigma)_{L_p} \leqslant M_6 \int_0^\sigma \frac{\widetilde{w}(f, 0, \xi)}{\xi} \, d\xi + M_9 \sigma \int_0^l \frac{\widetilde{w}(f, 0, \xi)}{\xi^2} \, d\xi + M_{10} \widetilde{w}(f, \sigma, \sigma)_{L_p} + \Omega_2(f, \sigma), \tag{8}$$

hold, where

$$\Omega_1(f) = \begin{cases} 0, & \text{if } \|\tilde{f}\|_{L_p} \geqslant l, \ p \geqslant 1, \\ \widetilde{M}_7 \min \left\{ \|\tilde{f}\|_{L_p} \ln \dfrac{1 + \|\tilde{f}\|_{L_p}}{\|\tilde{f}\|_{L_p}}; \ (\|\tilde{f}\|_{L_p})^{1-1/p} \right\}, & \text{for } p > 1, \\ \overline{M}_8 \|\tilde{f}\|_{L_p} \ln \dfrac{1 + \|\tilde{f}\|_{L_p}}{\|\tilde{f}\|_{L_p}}, & \text{for } p = 1, \end{cases}$$

$$\Omega_2(f) = \begin{cases} M_7 \min\left\{\widetilde{w}(f,\sigma,\sigma)\ln\dfrac{1+\sigma}{\sigma},\ \widetilde{w}(f,\sigma,\sigma)\sigma^{-1/p}\right\}, & \text{for } p>1,\\ M_8\widetilde{w}(f,\sigma)\ln\dfrac{1+\sigma}{\sigma}, & \text{for } p=1.\end{cases}$$

Definition 2. A function $u(t) \in H^p_\sigma(N, N_1)$ if it satisfies conditions analogous to (b_1), (b_2).

Suppose that, for any $u(t) \in H^p_\sigma(N, N_1)$ and $t, t_0 \in \Gamma\times\Gamma$, the function $K(t_0, t, u)$ satisfies the conditions:

$$\left\{\int_0^l |K(t(s), t(s+h), u(t(s))) - K(t(s), u(t(s)))|^p\, ds\right\}^{1/p} \leqslant M_{11}h^\alpha, \quad (9)$$

$$\sup_{0<\xi<l}\left\{\int_0^l |K(t(s+h), t(s+\xi), u(t(s+\xi))) - K[t(s), t(s+\xi), u(t(s+\xi))]|^p\ ds\right\}^{1/p} \leqslant M_{12}h^\alpha \ln^{-1}\left(1+\frac{1}{h}\right); \quad (10)$$

$$\left\{\int_0^l |K(t(s), t(s), u(t(s))) - K(t(s), t(s), v(t(s)))|^p\, ds\right\} \leqslant M_{13}\left\{\int_L |u(t) - v(t)|^p\, ds\right\}^{1/p} \quad (11)$$

or conditions (9), (11) and

$$\left\{\int_0^l |K[t(s+h), t(s), u(t(s))] - K[t(s), t(s), u(t(s))]|^p\, ds\right\}^{1/p} \leqslant \overline{M}_{10}h^{\alpha_1}, \quad (10')$$

$$\left\{\int_0^l\int_0^l |K[t(s+h), t(s+\xi), u(t(s+\xi))) - K(t(s), t(s+\xi), u(t(s+\xi))|^p\, d\xi\right\}^{1/p} \leqslant M_{10}h^{\alpha_2}. \quad (10'')$$

From inequalities (7), (8) it follows that

Theorem 3. *If the function $K(t_0, t, u)$ satisfies conditions (9), (10), (11), or conditions (9), (10′), (11), (10″), and $\alpha_2 - 1/p > 0$, then for $|\lambda| \leq M_2$, where M_2 is sufficiently small, the operator (a_2) acts in $H^p_\delta(N, N_1)$ continuously in the sense of the metric of the space L_p, where $\delta = \min\{\alpha; \alpha_1; (\alpha_2 - 1/p)\}$.*

By the Schauder topological principle, from this theorem there follows the existence of a solution of equation (a_2^0) in $H_\delta(N, N_1)$.

Remark 3. One can give examples of essentially "bad" functions $K(t_0, t, u)$ satisfying conditions (9), (10), (11), but not satisfying conditions (10′), (10″), and there are functions satisfying conditions (9), (10′), (10″), but not satisfying condition (10).

Denote

$$g(t, \tau, u) = K(t, \tau, u) - K(\tau, \tau, u).$$

Suppose that for all $u(t), v(t) \in H_\delta^0$ one has

$$\left\{ \int_L \left| \int_L \frac{g(t, \tau, u(\tau)) - g(t, \tau, v(\tau))}{\tau - t}\, d\tau \right|^{p_1} |dt| \right\}^{1/p_1} \leq$$

$$\leq M_{13} \left\{ \int_L |u(t) - v(t)|^{p_1} |dt| \right\}^{1/p_1} \tag{12}$$

at least for some $P_1 > 1$. This condition is satisfied, for example, by all functions

$$K(t, \tau, u) \equiv R(t, \tau) Q(t, u),$$

if

$$\left\{ \int_0^l |R(t(s+h), t(s+h)) - R(t(s), t(s))|^{p_1} \, ds \right\}^{1/p_1} \leq M_{14} h^\alpha,$$

$$\left\{ \int_L |Q(t, u(t)) - Q(t, v(t))|^{p_1} |dt| \right\}^{1/p_1} \leq M_{15} \left(\int_L |u(t) - v(t)|^{p_1} \right)^{1/p_1}$$

for arbitrary $u(t), v(t) \in L_p(\Gamma)$ and $\alpha > 1/p_1,\ p_1 \geq 2$.

Theorem 4. *If $K(t, t, u(t)) \in L_{p_1}(\Gamma_1)$ and $K(t, \tau, u)$ satisfies condition (12), then for all $|\lambda| \leq M_3$, where M_3 is a certain constant, the operator (a_2^c) will be a continuous contraction operator in L_{p_1}.*

Thus, under the assumptions made, equation (a_2^c) has a unique solution in L_{p_1}; this solution can be found by Picard' s method of successive approximations, and the successive approximations will converge in the sense of the metric of the space L_{p_1}. But if, in addition, $K(t, \tau, u)$ satisfies conditions (9), (10), (11), or conditions (9), (9″), (10″), (11), then this unique solution will lie in some $H_\varphi^p(\bar{N}_1, N)$, and it can be found by the method of successive approximations if as the initial function $u(t_0)$ one takes a function from $H^p_\varphi(\bar{N}_1, N)$, with the successive approximations converging in the sense of the metric L_p, even if $p > p_1$.

Remark 4. We shall say that $u(t) \in H^p_\varphi(N, N_1)$ if $\|u\|_{L^p} \leq N$, $w(u', \sigma)_{L^p} \leq N_1\varphi_1(\sigma)$; correspondingly we also define $H^p_{\varphi_1,\varphi_2}(N, N_1, N_2)$. If one takes $\varphi(\sigma) \in \Phi$ (for the definition of Φ, see, for example, in ([4])), then under corresponding restrictions on the function $K(t_0, t, u)$, Theorems 1, 2, and 3 also hold for the sets $H^p_{\varphi_1,\varphi_2}(N, N_1, N_2)$ and $H^p_\varphi(N, N_1)$.

Remark 5. Inequalities (2), (6), (8) in a certain special case coincide with the corresponding inequality of paper ([5]).

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Note: Figure translations are in progress. See original paper for figures.