ON A COMPACTIFICATION OF NONCOMPACT SYMMETRIC RIEMANNIAN SPACES

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Abstract

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MATHEMATICS

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ON A COMPACTIFICATION OF NONCOMPACT SYMMETRIC RIEMANNIAN SPACES

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1. Let $\mathcal{E}$ be a noncompact symmetric Riemannian space of rank l, with maximal connected group of motions G and stationary subgroup U of some point x_0 of it.

Denote the Lie algebras of the groups G and U by $\mathfrak{G}$ and $\mathfrak{U}$; denote by $\mathcal{E}\langle\cdot,\cdot\rangle$ the invariant scalar product in the algebra $\mathfrak{G}$ that gives rise to the Riemannian metric. The orthogonal complement $\mathcal{L}$ of the subalgebra $\mathfrak{U}$ in the algebra $\mathfrak{G}$ is identified, as usual, with the tangent space of the space $\mathcal{E}$ at the point x_0.

Let $\mathfrak{h}$ be some Cartan subalgebra of the space $\mathcal{E}$ contained in $\mathcal{L}$, $\Sigma = \Sigma(\mathfrak{h})$ its system of roots, and Z the set of integers. The set

$$I = I(\mathfrak{h}) = \left\{ \mu \in \mathfrak{h} \;\middle|\; \frac{2\langle\mu,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in Z \text{ for all } \alpha \in \Sigma \right\} \tag{1}$$

will be called the lattice of the space $\mathcal{E}$.

It is obvious that any two lattices of the space $\mathcal{E}$ are conjugate with respect to the adjoint group $\mathrm{Ad}(G)$ of the group G.

Let $(\mathfrak{h}_0, \dots, \mathfrak{h}_{k-1})$, where $1 \le k \le l$, be an orthonormal root of pairwise commuting vectors of the space $\mathcal{L}$. Choose some Cartan subalgebra $\mathfrak{h}$ of the space $\mathcal{E}$ so that

$$\mathfrak{h}_0, \dots, \mathfrak{h}_{k-1} \in \mathfrak{h}\mathcal{L}, \tag{2}$$

and select from the system of roots $\Sigma = \Sigma(\mathfrak{h})$ and the lattice $I = I(\mathfrak{h})$ the subsystem

$$\Sigma^{(k)} = \Sigma(\mathfrak{h}_0, \dots, \mathfrak{h}_{k-1}) = \{\alpha \in \Sigma \mid \langle\alpha, \mathfrak{h}_s\rangle = 0 \text{ for all } 0 \le s \le k-1\} \tag{3}$$

and the sublattice

$$I^{(k)} = I(\mathfrak{h}_0, \dots, \mathfrak{h}_{k-1}) = \{\mu \in I \mid \langle \mu, \mathfrak{h}_s \rangle = 0 \text{ for all } 0 \leq s \leq k-1\}. \qquad (4)$$

We shall call the tuple $(\mathfrak{h}_0, \dots, \mathfrak{h}_{k-1})$ admissible if, for every s $(1 \leq s \leq k-1)$, $\mathfrak{h}_s \in ((I(\mathfrak{h}_0, \dots, \mathfrak{h}_{k-1})))$ (the linear span of I).

2. To the space $\mathcal{E}$ with fixed point x_0, by means of its root system Σ and lattice I, one can associate a family of symmetric Riemannian spaces, which we shall call the family of boundary spaces, or ε-spaces, of the space $\mathcal{E}$.

An arbitrary ε-space of the space $\mathcal{E}$ is constructed as follows. Let $(\mathfrak{h}_0, \dots, \mathfrak{h}_{k-1})$, where $1 \leq k \leq l$, be an arbitrary admissible tuple; let $\mathfrak{h}$ be some Cartan subalgebra of the space $\mathcal{E}$ satisfying (2); let $G^{(k)} = G(x_0, \mathfrak{h}_0, \dots, \mathfrak{h}_{k-1})$ be the noncompact semisimple Lie group generated by the root subsystem (3); $U^{(k)} = U(x_0, \mathfrak{h}_0, \dots, \mathfrak{h}_{k-1}) = G^{(k)} \cap U$, and $H_{(k)} = H_{(k)}(x_0, \mathfrak{h}_0, \dots, \mathfrak{h}_{k-1})$ the connected noncompact commutative group corresponding to the orthogonal complement $\mathfrak{h}_{(k)} = \mathfrak{h}(x_0, \mathfrak{h}_0, \dots, \mathfrak{h}_{k-1})$ of the space $((\Sigma^{(k)}))$ in the space $((I^{(k)}))$ $(^{3,4})$.

The direct product $\widehat{G}^{(k)}$ of the groups $G^{(k)}$ and $H_{(k)}$

$$\widehat{G}^{(k)} = \widehat{G}(x_0, \mathfrak{h}_0, \dots, \mathfrak{h}_{k-1}) = G^{(k)} \cdot H_{(k)} \qquad (5)$$

does not depend on the choice of the Cartan subalgebra $\mathfrak{h}$ satisfying (2), and gives rise to the symmetric Riemannian space

$$S^{(k)} = S(x_0, \mathfrak{h}_0, \dots, \mathfrak{h}_{k-1}) = \widehat{G}^{(k)} / U^{(k)} \quad (1 \leq k \leq l), \qquad (6)$$

which is called the ε-space of the k-th degree of the space $\mathcal{E}$, corresponding to its admissible tuple $(\mathfrak{h}_0, \dots, \mathfrak{h}_{k-1})$.

The set of all ε-spaces (6), supplemented for generality of the discussion by the space

$$S^{(0)} = \mathcal{E}, \qquad (7)$$

is the family of ε-spaces of the space $\mathcal{E}$.

Every ε-space $S^{(k)}$ which does not degenerate into a point has non-positive curvature and is homeomorphic to Euclidean space. As a Riemannian space, it decomposes into the direct product

$$S^{(k)} = \mathcal{E}^{(k)} \times E^{(k)} \qquad (8)$$

of the noncompact symmetric space

$$\mathcal{E}^{(k)} = \mathcal{E}(x_0, \mathfrak{h}_0, \dots, \mathfrak{h}_{k-1}) = G^{(k)} / U^{(k)}$$

and the Euclidean space

$$E^{(k)} = E(x_0, \mathfrak{h}_0, \dots, \mathfrak{h}_{k-1}) = H_{(k)}/\{e\}$$

(e is the identity element of the group G).

To each k-pencil $F^{(k)}$ of geodesic ε-spaces of

$$S^{(k)} = S(x_0, \mathfrak{h}_0, \dots, \mathfrak{h}_{k-1})$$

(^{1}) there corresponds, in a one-to-one manner, the ε-space

$$S^{(k)}(x_0^{(k)}, F^{(k)}) = S(x_0, \mathfrak{h}_0, \dots, \mathfrak{h}_{k-1}, \mathfrak{h}_k), \tag{9}$$

where $\mathfrak{h}_k$ is the tangent vector with which the geodesic $\gamma^{(k)} \in F^{(k)}$ passes through the point $x_0^{(k)}$ with stationary subgroup $U^{(k)}$. The space (9) is called the ε-space of the k-pencil $F^{(k)}$ of geodesics of the space $S^{(k)}$.

3. From the set of all projective representations of the motion group G of the space $\mathcal{E}$, one can single out a subset θ of its so-called admissible representations, or ∂-representations. A real projective representation φ of the group G is called its ∂-representation if in its space P_φ there exists a point $\varphi(x_0)$ satisfying the conditions:

$$U \subseteq G(\varphi, x_0), \qquad P_\varphi = P(\varphi, x_0), \tag{10}$$

where $G(\varphi, x_0)$ is the stationary subgroup of the point $\varphi(x_0)$ in the group G, and $P(\varphi, x_0)$ is the projective linear manifold of the space P_φ generated by the orbit $\mathcal{E}_\varphi$ of the group G passing through the point $\varphi(x_0)$. We note that the space $\mathcal{E}_\varphi$ is homeomorphic to the space $\mathcal{E}$. If the ∂-representation φ satisfies the equality $U = G(\varphi, x_0)$, then the space $\mathcal{E}_\varphi$ is isomorphic to the space $\mathcal{E}$ and is called a projective representation of the space $\mathcal{E}$, while its closure $\overline{\mathcal{E}}_\varphi$ in the space P_φ is called a projective compactification (^{2,3}).

4. Let

$$P_\theta - \prod_{\varphi \in \theta} P_\varphi$$

be the direct product of the spaces of all ∂-representations of the motion group G of the space $\mathcal{E}$, endowed with the Tikhonov topology. It is obvious that P_θ is a compact metrizable space.

Let $p = (p_\varphi)_{\varphi \in \theta}$ be an arbitrary point of the space P_θ, and let

$$\theta(x_0) = (\varphi(x_0))_{\varphi \in \theta}$$

(see 3). It is not hard to show that the formula

$$\theta(g)p = (\varphi(g)p_\varphi)_{\varphi\in\theta} \quad (p \in P_\theta,\ g \in G) \tag{11}$$

turns the group G into a topological group of transformations of the space P_θ, in which the stationary subgroup $G(\theta, x_0)$ of the point $\theta(x_0)$ coincides

with stationary subgroup U of the point x_0. It follows from this that the orbit $\mathcal{E}_\theta$ of the group G, passing through the point $\theta(x_0)$, is a representation of the space $\mathcal{E}$, and its closure $\overline{\mathcal{E}}_\theta$ in the space P_θ is a compactification of this space.

For any e-space (6) and its factors (8) one can also construct representations and compactifications in the space P_θ. This is done as follows. In the space P_θ, by means of a chain of limiting transitions,

$$\theta(x_0') = \lim_{t\to+\infty} \theta(\exp \mathfrak{h}_0 t)\theta(x_0), \dots, \theta(x_0^{(k)}) = \lim_{t\to\infty} \theta(\exp \mathfrak{h}_{k-1} t)\theta(x_0^{(k-1)})$$

one constructs the point $\theta(x_0^{(k)})$. The orbits $S_\theta^{(k)}, \mathcal{E}_\theta^{(k)}, E_\theta^{(k)}$ passing through it of the groups $\overline{G}^{(k)}, G^{(k)}, H_{(k)}$ are isomorphic to the spaces $S^{(k)}, \mathcal{E}^{(k)}, E^{(k)}$ and are called their representations generated by the compactification $\overline{\mathcal{E}}_\theta$; and the closures $\overline{S}_\theta^{(k)}, \overline{\mathcal{E}}_\theta^{(k)}, \overline{E}_\theta^{(k)}$ of these orbits in the space P_θ are compactifications of these spaces, also generated by the compactification $\overline{\mathcal{E}}_\theta$.

Let us describe the structure of the boundary $\mathcal{T}_\theta = \overline{\mathcal{E}}_\theta \setminus \mathcal{E}_\theta$ of the compactification $\overline{\mathcal{E}}_\theta$ and some of its connections with the geometry of the space $\mathcal{E}$. Let $S^{(k)}$ $(k \geq 0)$ be an arbitrary e-space of the space $\mathcal{E}$ not degenerating to a point (^{6,7}); let $\gamma^{(k)}$ be a geodesic of the space $S^{(k)}$; let $\mathfrak{h}_k$ be the tangent vector of the geodesic $\gamma^{(k)}$ at some point $x_0^{(k)}$, and let $\exp \mathfrak{h}_k t$ $(-\infty < t < +\infty)$ be the group of translations of the space $S^{(k)}$ along the geodesic $\gamma^{(k)}$. The point

$$\theta(x_{\gamma(k)}) = \lim_{t\to+\infty} \theta(\exp \mathfrak{h}_k t)\theta(x_0^{(k)}) \tag{12}$$

is naturally called the improper point of the geodesic $\gamma^{(k)}$ in the compactification $\overline{S}_\theta^{(k)}$.

Theorem 1. *All geodesics belonging to one and the same $\mathfrak{h}$-bundle of the e-space $S^{(k)}$ (1) have, in the compactification $\overline{S}_\theta^{(k)}$, a common improper point.*

Let $F^{(k)}$ be an arbitrary k-bundle of geodesics of the space $S^{(k)}$ (1), and let $S^{(k)}(x_0^{(k)}, F^{(k)})$ be the corresponding e-space (9). We shall call the representation $S_\theta^{(k)}(x_0^{(k)}, F^{(k)})$ of the space $S^{(k)}(x_0^{(k)}, F^{(k)})$, generated by the compactification $\overline{\mathcal{E}}_\theta$, the component of the boundary corresponding to the k-bundle $F^{(k)}$.

Theorem 2. *The boundary component*

$$S_\theta^{(k)}(F^{(k)}) = S_\theta^{(k)}(x_0^{(k)}, F^{(k)})$$

of any k-bundle $F^{(k)}$ of geodesics of the space $S^{(k)}$ does not depend on the choice of the initial point $x_0^{(k)}$ and coincides with the set of improper points of all geodesics of this k-bundle in the compactification $\overline{S}_\theta^{(k)}$. It decomposes into the direct product

$$S_\theta^{(k)}(F^{(k)}) = \mathcal{E}_\theta^{(k)}(x_0^{(k)}, F^{(k)}) \times E_\theta^{(k)}(x_0^{(k)}, F^{(k)}) \tag{13}$$

of representations of the noncompact and Euclidean factors of the e-space $S^{(k)}(x_0^{(k)}, F^{(k)})$, and its foliations, generated by this decomposition, do not depend on the choice of the initial point $x_0^{(k)}$.

Let $\Phi^{(k)}$ be the set of all k-bundles of the e-space $S^{(k)}$ $(k \geq 0)$, and let $\mathcal{T}_\theta^{(k)} = \overline{S}_\theta^{(k)} / S_\theta^{(k)}$.

Theorem 3. *For any e-space $S^{(k)}$ $(k \geq 0)$ not degenerating to a point,*

$$\mathcal{T}_0^{(k)} = \bigcup_{F^{(k)} \in \Phi^{(k)}} \overline{S}_\theta^{(k)}(F^{(k)}), \tag{14}$$

where

$$\overline{S}_\theta^{(k)}(F_1^{(k)}) \cap \overline{S}_\theta^{(k)}(F_2^{(k)}) = \emptyset \quad \text{for } F_1^{(k)} \neq F_2^{(k)}. \tag{15}$$

It follows directly from Theorem 3 that

Theorem 4. *The family of representations of all k-spaces of the space $\mathcal{E}$ of nonzero degrees (6), generated by the compactification $\mathcal{E}_\theta$, is a decomposition of its boundary $\mathcal{T}_\theta$.*

5. Let us compare the compactification $\mathcal{E}_\theta$ with the compactification $\overline{\mathcal{E}}$ of F. I. Karpelevich [1].

Theorem 5. *For any noncompact symmetric Riemannian space $\mathcal{E}$, the natural mapping $\theta^{-1} : \mathcal{E}_\theta \to \mathcal{E}$ can be extended to a continuous surjection $\overline{\theta}^{-1} : \overline{\mathcal{E}}_\theta \to \overline{\mathcal{E}}$.*

Theorem 6. *If the rank of the space $\mathcal{E}$ is greater than 1, then it is impossible to extend the natural mapping $\theta : \mathcal{E} \to \mathcal{E}_\theta$ to a continuous mapping $\overline{\theta} : \overline{\mathcal{E}} \to \overline{\mathcal{E}}_\theta$.*

Remark 1. For spaces $\mathcal{E}$ of rank 0 or 1, the compactifications $\mathcal{E}_\theta$ and $\overline{\mathcal{E}}$ are equivalent.

Remark 2. From Theorems 5 and 6 it is clear that the compactification $\overline{\mathcal{E}}_\theta$ has a more developed boundary than the compactification $\overline{\mathcal{E}}$.

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Note: Figure translations are in progress. See original paper for figures.

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