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MATHEMATICS

1970

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Abstract

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UDC 517.944

MATHEMATICS

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THE CANONICAL OPERATOR ON A LAGRANGIAN MANIFOLD WITH A COMPLEX GERM AND A REGULARIZER FOR PSEUDODIFFERENTIAL OPERATORS AND DIFFERENCE SCHEMES

(Presented by Academician A. N. Tikhonov on 27 VIII 1970)

For simplicity we begin with the consideration of the pseudodifferential equation

$$i\partial\psi/\partial t - L(\hat{p}, x, t)\psi = 0, \qquad x \in R^N, \qquad \psi|_{t=0} = \psi_0(x),$$

$$L(\hat{p}, x, t)\psi(x, t) = \frac{1}{(2\pi)^N}\int e^{ipx}L(p, x, t)\,dp\int e^{-ip\xi}\psi(\xi, t)\,d\xi, \tag{1}$$

where $L(p, x, t) \in C^\infty[R^{2N+1}]$ for $|p| \neq 0$ has the following properties.

1. There exists $\lim\limits_{h\to 0} L(hp, x, t) = L_0(x, t) \in C^\infty$; $L(hp, x, t)$ is a linear function of the variable h.
2. Let $\tilde{H}(p, x, t) \overset{\text{def}}{=} \operatorname{Im}[L(p, x, t) - L_0(x, t)] \leq 0$.

2a. Denote $H(p, x, t) = \operatorname{Re}[L(p, x, t) - L_0(x, t)]$.

3. There exists, in the whole, a solution of the system

$$\dot{x} = \partial H/\partial p, \qquad \dot{p} = -\partial H/\partial x, \qquad x = x^0, \qquad p = p^0 \neq 0. \tag{2}$$

Theorem 1. Under conditions 1)–3), the solution $\psi(x, t)$, continuous in t, of problem (1), for $\psi_0(x, 0) \in H_s[R^N]$, exists and is unique in $H_s[R^N]$.

Theorem 2. The solution of the Cauchy problem (1) will be infinitely differentiable for any function $\psi_0(x)$ belonging to C^∞ outside the set $\Omega \subset R^N$, at those and only those points (x, t) of the space R^{N+1} which do not belong to the union of the bicharacteristics $X(x^0, p^0, t)$ for all $x^0 \in \Omega$ and $|p^0| \neq 0$.

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Theorems 1 and 2 generalize Theorem 2 of paper ([1]) to the complex case and are proved also with the aid of constructing a regularizer in the whole. They are readily generalized to the case of a polynomial in $i\partial/\partial t$ on a smooth manifold M^N.

Now consider a difference scheme with $k+1$ layers in t and, generally speaking, infinitely many layers in $x \in R^N$. In contrast to the deep results obtained with the aid of energy estimates ([2,3]), here we construct a regularizer with the aid of the characteristics of difference schemes introduced by the author in ([4]).

Consider a one-parameter family of grids with step h in $x \in R^N$ and $\tau = \alpha h$ in t $(0 < h < 1)$. We smoothly extend functions on the grid u_m^n, where m is a multi-index, $m = (m_1, \dots, m_N)$, to functions $u^n(x)$ $(u^n(mh) = u_m^n)$. The operator $T_{x_l} u_m^n = u^n_{m_1,\dots,m_l+1,\dots,m_N}$ can be represented as $T_{x_l} u = e^{h\partial/\partial x_l} u$. Therefore any difference operator can be written as an operator of the form

$$Pu^n \overset{\text{def}}{=} \sum_{j=0}^{k} a_j(h\hat{p}, x, \tau(n+j), h)\, u^{n+j},$$

where $a_j(p, x, t, h) \in C^\infty[R^{2N+2}]$ for $|p| \neq 0$. Consider the difference problem

Cauchy

$$Pu^n = 0, \qquad u^i = f^i(x), \qquad i = 0, \dots, k-1. \tag{3}$$

We shall call the following function the principal symbol:

$$\rho(\lambda, p, x, t) = \sum_{j=0}^{k} a_j(p, x, t, 0)\lambda^j.$$

Assume that

1) All roots $\lambda_i(p, x, t)$, $i = 1, \dots, k$, of the equation $\rho(\lambda, p, x, t) = 0$ do not exceed unity in modulus.
2) In a neighborhood of the set $|\lambda_j| = 1$ the function

$$H^i(hp, x, t) \overset{\text{def}}{=} \frac{1}{\alpha} \arg \lambda_i(hp, x, t) \in C^\infty[R^{2N+2}]$$

for $|p| \neq 0$ and vanishes for $h = 0$, while the surfaces $H^i(hp, x, t) = 0$ do not intersect for $|p| \neq 0$ for any $0 \leq h \leq 1$.
3) The solution of the system

$$\dot{x} = \partial H^i/\partial p, \qquad \dot{p} = -\partial H^i/\partial x, \qquad i = 1, \dots, k, \tag{4}$$

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$$x(0) = x^0, \qquad p(0) = p^0 \neq 0$$

exists globally.

Definition. Let $X^i(x^0, p^0, t)$, $P^i(x^0, p^0, t)$ be the solution of problem (4) satisfying the condition

$$H^i(P^i, X^i, t) \stackrel{\text{def}}{=} \ln |\lambda_i(P^i(x^0, p^0, t), X^i(x^0, p^0, t), t)| = 0$$

for all $0 \leq t \leq T$. The curve $X^i(x^0, p^0, t)$, $0 \leq t \leq T$, will be called a bicharacteristic of the difference operator P.

Theorem 3. 1) Under conditions 1)–3) the difference scheme (3) is stable in L_2 on the grid in the sense that the solution $u(x,t)$ $(t = n\tau)$ of the problem

$$Pu = 0, \quad u^i = f^i(x), \quad i = 0, \dots, k-1,$$

satisfies the estimate

$$\|u(x,t)\|_{L_2} \leq M(t) \|f(x)\|_{L_2}, \qquad f(x) = (f^0(x), f^1(x), \dots, f^{k-1}(x)),$$

where $M(t)$ does not depend on h.

2) Any difference derivative of the solution of the Cauchy problem (3) will be bounded as $h \to 0$ for any function $\psi(x, \tau_i)$, $i = 0, \dots, k-1$, $x = mh$, any difference derivative of which is bounded as $h \to 0$ outside the set $\Omega \subset R^N$, at those and only those points (x,t) of the space R^{N+1} $(x = mh,\ t = n\tau)$ which do not belong to the closure of the union of the bicharacteristics $X^i(x^0, p^0, t)$ for all $i = 1, 2, \dots, k$, $x^0 \in \Omega$, $\pi \geq p_i^0 > 0$, $i = 1, \dots, k$.

This theorem is generalized to the case when the operator P is given on a manifold M^N, under the condition that it maps a function on the grid on this manifold to a function on the same grid (for example, when M^N is parallelizable).

The constructions of the regularizer for (1) and (3) are carried out by means of the construction of the canonical operator* (c.o.) on the Lagrangian manifold with complex germ introduced below.

Consider, in the $2N$-dimensional Euclidean real phase space p, q, an N-dimensional Lagrangian** manifold

$$\Lambda^N = \{q = x(\alpha),\ p = \xi(\alpha)\}, \quad \alpha \in R^N$$

with a measure $\sigma \in C^\infty$ on it. On this manifold consider a smooth vector field $y(\alpha), \eta(\alpha)$, vanishing on a submanifold Γ, such that on Γ the relations

$$\{y(\alpha), \eta(\alpha)\}_{\alpha \in \Gamma} = 0, \qquad \{\eta(\alpha), x(\alpha)\}_{\alpha \in \Gamma} = \{y(\alpha), \xi(\alpha)\}_{\alpha \in \Gamma}.$$

We shall call this field a complex germ. Let $F(\alpha)$ be such a function on Λ^N in a neighborhood of Γ that

$$dF = \eta\, dx - y\, d\xi + [O(y^2) +$$

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* We have transferred this name from the real case, although in the complex case the c.o. can no longer give a representation of the group of canonical transformations [4].

** That is, the Lagrange brackets $\{x(\alpha), \xi(\alpha)\}$ are equal to zero.

$+O(\eta^2)](|dx|+|d\xi|)$, nonnegative and vanishing on Γ, and $f(\alpha)$ is a function on Λ^N in a neighborhood of Γ such that $df = y\,d\eta + [O(y^2)+O(\eta^2)]|d\eta|$, vanishing on Γ. The collection $\{\Lambda^N, y(\alpha), \eta(\alpha), F(\alpha), f(\alpha)\}$ will be called a Lagrangian manifold with complex germ, or a μ-manifold. The manifold Λ^N can be covered by charts with coordinates

$$\beta_{i_1,\dots,i_k} = q_{i_1},\dots,q_{i_k},\ p_{i_{k+1}},\dots,p_{i_N}, \tag{5}$$

where $*$ $i_\nu \neq i_\mu$ for $\nu \neq \mu$. We define three zero-dimensional cochains with values in the sheaf of complex-valued functions.

1) In the chart $(u_j, \beta_{i_1,\dots,i_k})$ define the S-action

$$S(u_j,\alpha) = \int_{\alpha_0}^{\alpha} \xi(\alpha)\,dx(\alpha) + f(\alpha) - \sum_{\nu=1}^{N-k} \left[p_{i_{k+\nu}} x_{i_{k+\nu}}(\alpha) + \eta_{i_{k+\nu}}(\alpha) y_{i_{k+\nu}}(\alpha)\right],$$

where α^0 is a fixed point on Λ^N.

2) In the chart with coordinates $q_1,\dots,q_N$ (a nonsingular chart) define the μ-action

$$\mu(u_j,\alpha) = -\frac{1}{2}\left(y(\alpha), \left\|\frac{\partial(\xi_k(\alpha)+i\eta_k(\alpha))}{\partial\alpha_j}\right\| \cdot \left\|\frac{\partial(x_k(\alpha)+iy_k(\alpha))}{\partial\alpha_j}\right\|^{-1} y(\alpha)\right). \tag{6}$$

In the chart $(u_j; \beta_{i_1,\dots,i_k})$ define the μ-action by making in (6) the substitution $\xi_{i_{k+\nu}} \to x_{i_{k+\nu}},\ x_{i_{k+\nu}} \to \xi_{i_{k+\nu}},\ \nu = 1,\dots,N-k$.

3) Define the Jacobian

$$I_{1,2,\dots,N}(\alpha) \overset{\text{def}}{=} \det\left\|\frac{\partial(x_k(\alpha)+iy_k(\alpha))}{\partial\alpha_j}\right\| \times \frac{d\alpha_1,\dots,d\alpha_N}{d\sigma}$$

in a nonsingular chart. In an arbitrary chart $(u_j, \beta_{i_1,\dots,i_k})$ we replace $x_{i_{k+j}} \to \xi_{i_{k+j}},\ y_{i_{k+j}} \to \eta_{i_{k+j}},\ j = 1,\dots,N-k$. The argument of the Jacobian $-\infty < \gamma_{i_1,\dots,i_k} < \infty$ is defined uniquely by means of the following procedure.

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Consider, on the intersection of two charts $(u_j, \beta_{i_1,\dots,i_k})$ and $(u_j, \beta_{i_1,\dots,i_{k-n}})$, the corresponding Jacobians $I_{i_1,\dots,i_k}(\alpha)$ and $I_{i_1,\dots,i_{k-n}}(\alpha)$, and put

$$v_{i_{k-\nu}}(t) = (x_{i_{k-\nu}} + iy_{i_{k-\nu}})\cos t + (\xi_{i_{k-\nu}} + i\eta_{i_{k-\nu}})\sin t.$$

Replacing in $I_{i_1,\dots,i_k}(\alpha)$
$x_{i_{k-\nu}} + iy_{i_{k-\nu}} \to v_{i_{k-\nu}}$, $\nu = 0, \dots, m-1$, we obtain a Jacobian $I(\alpha, t)$, which for $t = 0$ equals $I_{i_1,\dots,i_k}(\alpha)$, and for $t = \pi/2$ equals $I_{i_1,\dots,i_{k-n}}(\alpha)$. We make the passage from 0 to $\pi/2$, bypassing the zeros of $I(\alpha, t)$ on the left in the complex t-plane (in the direction of increasing phase). Thus we have joined the phases of the Jacobians $\gamma_{i_1,\dots,i_k}$ and $\gamma_{i_1,\dots,i_{k-n}}$ in the two charts. Having defined, in a single-valued way, $-\pi < \arg I \leq \pi$, we obtain the definition of the path index $l[a^0, \alpha]$ by setting
$2\pi \operatorname{Ind} l[a^0, a^1] = \gamma(a^0) - \arg I(a^0) - [\gamma(a^1) - \arg I(a^1)]$, where a^0 and a belong to nonsingular charts. We now define the canonical operator.

Let $\varphi(x, \alpha)$ be a finite infinitely differentiable function of α, with support in the $(u_j, \beta_{i_1,\dots,i_k})$-chart and with values in some Hilbert space H.

* That is, each chart $(u_i; \beta_{i_1,\dots,i_k})$ is diffeomorphically projected onto one of the $N-k$-dimensional $(N \geq k \geq 0)$ momentum planes parallel to the remaining coordinate axes.

Let A be a self-adjoint operator in H having an inverse. Put

$$K_A^{a_0}\varphi(x,\alpha) = \Phi_A^{p_{i_{k+1}},\dots,p_{i_N}} \frac{\cos\gamma_{i_1,\dots,i_k} + i\sin\gamma_{i_1,\dots,i_k}\operatorname{sgn} A}{\sqrt{\left|I_{i_1,\dots,i_k}(\alpha)\right|}} \exp\{iA[S(u_j,\alpha)+$$

$$+\operatorname{Re}\mu(u_j,\alpha)] - |A|[F(\alpha) + \operatorname{Im}\mu(u_j,\alpha)]\}\varphi(x,\alpha),$$

where $\Phi_A^{p_{i_{k+1}},\dots,p_{i_N}}$ is the A-Fourier transform ([1,4]) with respect to the arguments $p_{i_{k+1}}, \dots, p_{i_N}$.

Let K_s be the Hilbert space with norm

$$\|f\|^2_{K_s} = \int \|(\sqrt{-\Delta + A^2q^2 + i})^s f\|^2_H \, dq.$$

The operator $K_A^{a_0}$ maps infinitely differentiable functions on Λ^N into the space K_s. We shall call it the canonical operator (c.o.).

Theorem 4. The c.o. $K_A^{a_0}$ in the factor space K_s/K_{s+1} is independent of the division into charts if and only if the eigenvalues λ of the operator A satisfy the relation

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$$\lambda \oint p\,dx = \frac{\pi}{2}\,\mathrm{Ind}\,\gamma$$

for any cycle on Γ, where $\mathrm{Ind}\,\gamma$ is the index of the path along this cycle.

In the case of interest to us, all $\oint p\,dx$ and $\mathrm{Ind}\,\gamma$ are equal to zero; therefore there are no conditions on the spectrum of A.

The Lagrangian manifold generated by system (2) is defined as the solution $x(a,t), \xi(a,t)$ of the problem

$$\dot{x} = \partial H(\xi,x,t)/\partial\xi, \qquad \dot{\xi} = -\partial H(\xi,x,t)/\partial x, \qquad x(0) = x^0, \quad \xi(0) = p^0,$$

where $|p^0| = 1$, and the complex germ $y(a,t), \eta(a,t)$ as the solution of the system

$$\dot{y}_k = \sum_i (\eta_i H_{\xi_k\xi_i} + y_i H_{\xi_k x_i}) + \widetilde{H}_{\xi_k}, \qquad k = 1,\dots,N,$$

$$\dot{\eta}_k = -\sum_i (\eta_i H_{x_k\xi_i} + y_i H_{x_k x_i}) - \widetilde{H}_{x_k}, \qquad y(a,0) = \eta(a,0) = 0.$$

Similarly to (1), using the c.o. with $H = L_2[\tau]$, $A = i\partial/\partial\tau$, a regularizer for problem (1) is constructed on the resulting Lagrangian manifolds with complex germ. The boundedness of the regularizer is proved by shifting backward in time along the same trajectories by a Weyl pseudodifferential operator with symbol $p_0 - H(p,x,t)$, which does not change the norm. In exactly the same way, for problem (3) a regularizer is constructed, where as the space H one takes the space of functions of two variables τ, h with norm

$$\|g\|_H^2 = \int_0^{h_0} dh \int_{-\pi/h}^{\pi/h} \left| \int e^{i\eta\tau} g(h,\tau)\,d\tau \right|^2 d\eta,$$

and as A, the operator $i\partial/\partial\tau$.

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Received
7 VIII 1970

References

1. V. P. Maslov, DAN, 177, No. 6 (1967).

2. A. A. Samarskii, Zhurn. vychisl. matem. i matem. fiz., 3, No. 2, 266, No. 3 (1963); 2, No. 4, 603, No. 5, 787, No. 6, 117 (1962).

3. P. Lax, L. Nirenberg, Matem., 1, 1 (1957).

4. V. P. Maslov, *Perturbation Theory and Asymptotic Methods*, Moscow, 1965.

Note: Figure translations are in progress. See original paper for figures.

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