HOMOGENEOUS GENERALIZED FUNCTIONS AND THE RADON TRANSFORM IN THE SPACE OF RECTANGULAR MATRICES OVER A NON-DISCRETE LOCALLY COMPACT DISCONNECTED FIELD

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HOMOGENEOUS GENERALIZED FUNCTIONS AND THE RADON TRANSFORM IN THE SPACE OF RECTANGULAR MATRICES OVER A NON-DISCRETE LOCALLY COMPACT DISCONNECTED FIELD

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1. Let $M_{n,m} = M_{n,m}(K)$ be the space of rectangular $m \times n$-matrices $x = x_{(ij)}$, $n \geq m$, with entries from a non-discrete locally compact disconnected field K. Denote by $S_{n,m}$ the space of finite, locally constant functions f on $M_{n,m}$. In other words, $S_{n,m}$ is the set of functions such that: 1) $f(x) = 0$ when $|x|$ is sufficiently large, $|x| = \max_j(|x_{ij}|)$; 2) for every $x \in M_{n,m}$ there exists an integer $k(x)$ such that $f(x + x_0) = f(x)$, $|x_0| \leq q^{-k(x)*}$. Here q is the number of elements of the residue field $\mathfrak{K} = \mathcal{O}/\mathfrak{P}$, where $\mathcal{O}$ is the subring of integral elements x of the field K, i.e., elements for which $|x| \leq 1$; $\mathfrak{P}$ is the maximal ideal in $\mathcal{O}$. A natural topology is introduced in $S_{n,m}$ ([1]), with respect to which $S_{n,m}$ is a complete linear space. By generalized functions on $M_{n,m}$ we shall henceforth mean linear continuous functionals on $S_{n,m}$.

Denote by $D_{n,m}(x)$ the maximum of the norms of the minors of order m of the matrix x^{**}. For every complex number λ, $\operatorname{Re}\lambda > 0$, define the generalized function $D^\lambda_{n,m}(x)$ by the convergent integral

$$(D^\lambda_{n,m}(x), f(x)) = \int_{M_{n,m}} D^\lambda_{n,m}(x) f(x)\, dx, \qquad f \in S_{n,m}.$$

For $\operatorname{Re}\lambda < 0$ define $D^\lambda_{n,m}(x)$ by analytic continuation. For any λ, the function $D^\lambda_{n,m}(x)$ is a spherically symmetric homogeneous generalized function of degree of homogeneity λ in the sense of the definition given below (see item 2).

The paper studies the function $D^\lambda_{n,m}$ as an analytic function of λ. The singular points of this function are found (Theorem 1), the residues at these singular points are computed (Theorem 2), the uniqueness of a spherically symmetric homogeneous generalized function is proved (Theorem 3), and the Fourier transform of the function $D^\lambda_{n,m}(x)$ is computed. The results are applied to the solution of a problem of integral geometry in the space $M_{n,m}$ $(n > m)$ (Theorem 5). (The analogous problem of integral geometry for the spaces $M_{n,m}(\mathbf{R})$ and $M_{n,m}(\mathbf{C})$ was solved in ([2]).) The resulting formula contains, as a special case for $m = 1$, the inversion formula for the Radon transform in affine space over the field K.

In what follows, the field K is assumed to have characteristic different from 2 and not to contain the field of dyadic numbers $\mathbf{Q}_2$.

* In view of condition 1), $k(x)$ can always be chosen independently of x.

** In particular, for $n = m$, $D_{n,m}(x) = D_n(x) = |\det x|$.

2. In this section we shall study the generalized function $D^\lambda_{n,m}$.

Theorem 1. *The generalized function $D^\lambda_{n,m}$, considered as an analytic function of λ, is regular for all λ except the points $\lambda = -n + i,\ i = 0, 1, \dots, m-1$, at which it has simple poles.*

We give formulas for the residues of $D^\lambda_{n,m}$ at the exceptional points. Let us first introduce the necessary notation. Let

$$M^r_{n,m} = \{x \in M_{n,m} \mid \operatorname{rang} x \le r\}.$$

We introduce a system of coordinates in $M^r_{n,m}$. Denote by y the matrix consisting of the first r columns of the matrix $x \in M^r_{n,m}$, and by y' the matrix consisting of the remaining $m - r$ columns of the matrix x. Then in general position $\operatorname{rang} x = r$ and $y' = y\mu$, where μ is an $r \times (m - r)$-matrix. We shall take the elements of the matrices y and μ as coordinates on $M^r_{n,m}$; functions on $M^r_{n,m}$ will be written in the form $f(x) = f(y, y\mu)$.

Theorem 2. *The residue of the generalized function $D^\lambda_{n,m}$ at the exceptional point $\lambda = -n + r,\ r = 0, 1, \dots, m - 1$, is a generalized function concentrated on the manifold $M^r_{n,m}$ and given by the formulas:*

$$\left(\operatorname*{Res}_{\lambda=-n} D^\lambda_{n,m}(x),\ f(x)\right) = \frac{\Phi(-n, m)}{\Phi(1, m-1)\ln q}\, f(0);$$

for $i = 1, 2, \dots, m - 1$

$$\left(\operatorname*{Res}_{\lambda=-n+i} D^\lambda_{n,m}(x),\ f(x)\right)$$
$$= \frac{\Phi(-n, m)\Phi(-m, i)(\ln q)^{-1}}{\Phi(-i, i)\Phi(1, m-i-1)\Phi(-n, i)} \int f(y, y\mu)\, D^{-n+m}_{n,i}(y)\, dy\, d\mu\ *,$$

where

$$dy = \prod_{i,j=1}^{n,r} dy_{ij}, \qquad d\mu = \prod_{s,t=1}^{r,i,m-r} d\mu_{st},$$

$$\Phi(\lambda,k)=(1-q^{\lambda})(1-q^{\lambda+1})\cdots(1-q^{\lambda+k-1}),\qquad \Phi(\lambda,0)=1;$$

$dy_{ij}, d\mu_{st}$ are measures invariant with respect to addition on K.

We shall call a generalized function φ on $M_{n,m}$ homogeneous of degree λ if it satisfies the condition

$$(\varphi(x), f(xa^{-1}))=|\det a|^{\lambda+n}(\varphi(x), f(x)),\qquad a\in GL(m,K).$$

Denote

$$M_{n,m}(\mathcal{O})=\{x\in M_{n,m}\mid x_{ij}\in\mathcal{O}\},\qquad U_m=\{u\in M_{n,m}(\mathcal{O})\mid |\det u|=1\}.$$

We shall call the homogeneous generalized function φ spherically symmetric if

$$(\varphi(x), f(ux))=(\varphi(x), f(x)),\qquad u\in U_n.$$

For $\lambda\neq -n+i,\ i=0,1,\dots,m-1$, the generalized function $D^{\lambda}_{n,m}$ is, obviously, a spherically symmetric homogeneous generalized function; for $\lambda=-n+i,\ i=0,1,\dots,m-1$, such is the residue of $D^{\lambda}_{n,m}$ at the corresponding points. Thus, for every complex number λ, we have constructed a spherically symmetric homogeneous generalized function of homogeneity degree λ. It is convenient to pass from the functions $D^{\lambda}_{n,m}(x)$ to the functions

$$F_{\lambda}(x)=\Phi(-\lambda-n,m)D^{\lambda}_{n,m}(x).$$

From what was said above it follows that $F_{\lambda}(x)$, as a function of λ, has neither singularities nor zeros in any finite domain. In particular, it is easy to see that

$$F_{-n}(x)=\Phi(-n,m)\delta(x). \tag{1}$$

Theorem 3. *For every complex number λ on $M_{n,m}$ there exists, up to a constant factor, only one generalized spherically symmetric homogeneous function of homogeneity degree λ, namely $F_{\lambda}(x)$.*

3. We define the Fourier transform of functions $f\in S_{n,m}$ by the formula

$$\tilde f(\xi)=\int_{M_{n,m}} f(x)\chi(\operatorname{Sp}^{t}\xi\cdot x)\,dx,$$

* On the basis of Theorem 1, $D^{\lambda}_{n,i}$ is regular at $\lambda=-n+m$.

where $\chi(x)$ is an additive character of rank 0 (1) on K (the symbol t denotes transposition).

Let us note the obvious properties of the Fourier transform: 1) the function f is recovered from its Fourier transform $\tilde f$ by the formula

$$f(x)=\int_{M_{n,m}}\tilde f(\xi)\chi(-\operatorname{Sp}{}^{t}\xi\cdot x)d\xi;$$

2) the Fourier transform is a bijective mapping of $S_{n,m}$ onto itself; 3) under the Fourier transform the characteristic function of the set $M_{n,m}(\mathcal{O})$ goes into itself. The Fourier transform $\tilde{\varphi}$ of a generalized function φ, as usual, is defined by the formula $(\tilde{\varphi}, \tilde{f}) = (\varphi, f(-x))$ for any $f \in S_{n,m}$.

Theorem 4. *The Fourier transform of the generalized function $F_\lambda(x)$ is given by the formula $\tilde{F}_\lambda(\xi) = F_{-\lambda-n}(\xi)$.*

4. Let us apply the result obtained to the solution of the following problem of integral geometry in the space $M_{n,m}$, $n > m$. We shall call a plane in $M_{n,m}$ a variety given by the linear matrix equation

$$(\xi, x) = \xi \cdot x = s, \tag{2}$$

where ξ is an $m \times n$-matrix of maximal rank. Clearly, the dimension of such a plane is $m(n-m)$; the dimension of the variety of all planes of the form (2) is mn, i.e. it has the same dimension as $M_{n,m}$. To each function $f \in S_{n,m}$ we put in correspondence its integrals over all possible planes (2) by the formula

$$\check{f}(\xi, s) = \int_{M_{n,m}} f(x)\delta(s - \xi \cdot x)\, dx, \tag{3}$$

where δ is the delta function on $M_{m,m} = M_m \simeq K^{m^2}$, and dx is the volume element on $M_{n,m}$. The expression for $\check{f}(\xi, s)$ can be written directly in the form of an integral over the plane (2):

$$\check{f}(\xi, s) = \int_{\xi \cdot x = s} f(x)\, \omega_\xi,$$

where δ is the delta function on $M_{m,m} = M_m \simeq K^{m^2}$, dx is the volume element on $= dx$*. It is not difficult to write an expression for ω_ξ in terms of the elements of the matrix x: $\omega_\xi = (-1)^{m(|\nu|-\rho)}|\det \xi_\nu|^{-m} dx^\nu$, where $\nu = (i^1, \dots, i^m)$ is an ordered set of numbers from the set $\{1, 2, \dots, n\}$; $|\nu| = i^1 + \dots + i^m$; ξ_ν is the matrix formed from the $i^1, \dots, i^m$-th columns of the matrix ξ; x^ν is the matrix obtained from the matrix x by deleting the $i^1, \dots, i^m$-th rows; $\rho = m^2(n - mn + m^2 + 1)/2$.

We shall call the function $\check{f}(\xi, s)$ the Radon transform of the function $f(x)$. The problem consists in obtaining an inversion formula for the Radon transform (3). For what follows let us note one property of the Radon transform: if $f_{x_0}(x) = f(x + x_0)$, then $\check{f}_{x_0}(\xi, s) = \check{f}(\xi, s + \xi \cdot x_0)$.

5. Introduce the variety $U_{n,m} = \{u \in M_{n,m}(\mathcal{O}) \mid D_{n,m}(u) = 1\}$. It is easy to see that $U_{n,m}$, along with $M_{n,m}(\mathcal{O})$, is an open compact set in $M_{n,m}$.

Theorem 5. *Let $\check{f}(\xi, s)$ be the Radon transform of a function $f \in S_{n,m}$. Then the inversion formula holds*

$$f(x) = \frac{\Phi(n-m, m)}{\operatorname{mes} U_m \Phi(-n, m)} \int_{U_{m,n}} \int_{M_m} \check{f}(u, s - u \cdot x) D_m^{-n}(s)\, ds\, du, \tag{4}$$

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* That is, whatever the function $f \in S_{n,m}$, the measure ω_ξ is such that

$$\int_{M_m} ds \int_{\xi \cdot x = s} f(x) \omega_\xi = \int_{M_{n,m}} f(x)\, dx.$$

where $\operatorname{mes} U_m = \Phi(-m, m)$, $du = \prod_{i,j}^{n,m} du_{ij}$ is a measure on the manifold $U_{n,m}$.

The proof is based on the equality

$$\int\limits_{U_{m,n}} D_m^\lambda(u \cdot x)\, du = \frac{\operatorname{mes} U_m\, \Phi(-\lambda - n, m)}{\Phi(-\lambda - m, m)} D_{n,m}^\lambda(x),$$

valid for all λ for which the integral converges. In view of this relation we have

$$\frac{\operatorname{mes} U_m}{\Phi(-\lambda - m, m)} \int f(x) \Phi(-\lambda - n, m) D_{n,m}^\lambda(x)\, dx = \int f(x) D_m^\lambda(u \cdot x)\, du\, dx.$$

The integral on the right-hand side of this equality is easily expressed through the Radon transform, and we obtain

$$\frac{\operatorname{mes} U_m}{\Phi(-\lambda - m, m)} \int f(x) \Phi(-\lambda - n, m) D_{n,m}^\lambda(x)\, dx = \int \check{f}(u, s) D_m^\lambda(s)\, ds\, du. \tag{5}$$

Both integrals converge for $\operatorname{Re} \lambda > 0$ and in this domain are analytic functions of λ. Continue these functions to the domain $\operatorname{Re} \lambda < 0$ and take their values at $\lambda = -n$. On the basis of (1), the value at $\lambda = -n$ of the function standing in the left-hand side of equality (5) is $\Phi(-n, m) f(0)$; D_m^λ is regular at $\lambda = -n$. Thus we have obtained

$$f(0) = \frac{\Phi(n-m, m)}{\operatorname{mes} U_m\, \Phi(-n, m)} \int \check{f}(u, s) D_m^{-n}(s)\, ds\, du. \tag{6}$$

Now, to obtain the final formula (4), it remains only to apply formula (6) to the function $f_{x_0}(x) = f(x + x_0)$.

6. Theorem 6. In order that a function $\check{f}(\xi, s)$, $\xi \in M_{n,m} \setminus M_n^{m-1}$, be the Radon transform of some function $f \in S_{n,m}$, it is necessary and sufficient that the following conditions be satisfied: 1) $\check{f}(\xi, s)$ is a homogeneous function of ξ and s of degree of homogeneity $-m$, i.e. $\check{f}(a\xi, as) = |\det a|^{-m} \check{f}(\xi, s)$, $\forall a \in GL(m, K)$; 2) $\check{f}(\xi, s) = 0$ when $|s|\,|\xi|^{-1}$ is sufficiently large; 3) $\check{f}(\xi, s)$ is piecewise constant in the aggregate ξ, s, i.e., for any fixed ξ_0, s_0 we have $\check{f}(\xi_0, s_0) = \check{f}(\xi, s)$ when $|\xi - \xi_0|$ and $|s - s_0|$ are sufficiently small; 4) the integral $\int \check{f}(\xi, s)\, ds_i$, where $ds_i = ds_{i1} \ldots ds_{im}$, does not depend on the elements of the i-th row of the matrix ξ.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.