ON TOPOLOGICAL GROUPS

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Abstract

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ON TOPOLOGICAL GROUPS

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Let X_1 be a subset of a topological space X. A system of sets $\{O_\alpha : \alpha \in \Gamma\}$, where $X_1 \subset \operatorname{int} O_\alpha \subset X$, is called a character of the set X_1 in X, if for every open set $U \supset X_1$ there exists O_{α_0} ($\alpha_0 \in \Gamma$) such that $O_{\alpha_0} \subset U$. B. A. Pasynkov calls a topological group almost metrizable ([1]) if it contains a bicompactum of countable character.

Theorem.** *Let G be a separable topological group and let H be its Weil-complete almost metrizable normal subgroup. Then there exists a set $A \subset G$ such that $\pi(A) = G/H$, where $\pi : G \to G/H$ is the natural projection, and the restriction $\pi/A : A \to G/H$ is a perfect mapping.*

Proof. Let $\{O_i\}_{i=1}^{\infty}$ be a character in H of a bicompact set $B \subset H$; we may assume that B contains the identity element e of the group G. Choose a system of neighborhoods $\{U_i\}_{i=1}^{\infty}$ of the identity so that: a) $U_{n+1}^2 \subset U_n$; b) $U_n^{-1} = U_n$; c) $U_n \cap H \subset O_n$. Put

$$L = \bigcap_{i=1}^{\infty} U_i;$$

L is a closed subgroup of the group G, and therefore the subgroup

$$K = L \cap H = \left(\bigcap_{i=1}^{\infty} U_i\right) \cap H =$$

$$= \bigcap_{i=1}^{\infty} (U_i \cap H) \subset \bigcap_{i=1}^{\infty} O_i = B$$

is bicompact.

Lemma 1. *The system of sets $\{V_i\}_{i=1}^{\infty}$, where $V_i = U_i \cap H$, is a character of the set K in H.*

The proof of this lemma is left to the reader.

We shall denote the space of left cosets of the group G modulo the subgroup L by G/L, and the natural mapping $x \mapsto xL$ by

$$\omega : G \to G/L.$$

Every element $s \in G$ defines a mapping $s : G/L \to G/L$ by the formula

$$s(\omega(x)) = \omega(sx)$$

(sometimes we shall omit the parentheses and write $s\omega(x)$). It can be proved that there exists a metric ρ on G/L with the following properties: 1) the mapping $\omega : G \to (G/L, \rho)$ is continuous; 2) for $s \in G$ and $z_1, z_2 \in G/L$,

$$\rho(sz_1, sz_2) = \rho(z_1, z_2);$$

3) $\{\omega(U_n)\}_{n=1}^{\infty}$ is a base of neighborhoods of the point $\omega(e)$ in $(G/L, \rho)$.

Lemma 2. $\omega(H)$ *is a complete subspace of the metric space* $(G/L, \rho)$.

Proof of Lemma 2. Let $\xi = \{A\}$ be any Cauchy filter in $\omega(H)$, and let η be an ultrafilter in H whose image $\omega(\eta)$ majorizes ξ. We shall prove that η is a Cauchy filter. Let V be a neighborhood of the identity in H; there exists $U_{i_0} \in \{U_n\}_{n=1}^{\infty}$ such that $U_{i_0} \cap H \subset KV$. Since $\omega(U_{i_0+1})$ is a neighborhood of $\omega(e)$ in $(G/L, \rho)$, for some $h_0 \in H$ the set

$$h_0\omega(U_{i_0+1}) = \omega(h_0 U_{i_0+1})$$

contains an element of the filter ξ. Consequently,

$$\omega^{-1}\omega(h_0 U_{i_0+1}) \cap H = h_0 U_{i_0+1} L \cap H \in \eta.$$

However

$$h_0 U_{i_0+1} L \cap H \subset h_0 U_{i_0+1} \cap H \subset h_0(U_{i_0} \cap H) \subset h_0 KV,$$

and from the bicompactness of $h_0 K$ it follows that $h_0 K \subset$

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* $\operatorname{int} O_\alpha$ is the interior of the set O_α.

** This theorem adjoins a result of B. A. Pasynkov (([2]), Theorem 2).

$$\subset \bigcup_{i=1}^{p} h_0 k_i V, \quad \text{where } k_i \in K \ (i = 1, 2, \dots, p).$$

Hence,

$$h_0 KV \subset \left(\bigcup_{i=1}^{p} h_0 k_i V \right) \times V \subset \bigcup_{i=1}^{p} h_0 k_i V^2 \in \eta.$$

The ultrafilter η must contain at least one term of the union

$$\bigcup_{i=1}^{p} h_0 k_i V^2,$$

i.e. η is a Cauchy filter in H. Therefore it converges to some $h \in H$; $\omega(h)$ will be a point of contact of the Cauchy filter ξ, and hence ξ converges to $\omega(h)$. The lemma is proved. Since $\omega(xH) = x\omega(H)$, the sets $\omega(xH)$ are also complete in $(G/L, \rho)$.

Let us note that

$$G/L = \bigcup_{x \in G} \omega(xH) = \bigcup_{x \in G} x\omega(H).$$

Moreover, for $x_1, x_2 \in G$, either $x_1\omega(H) \cap x_2\omega(H) = \emptyset$, or $x_1\omega(H) \equiv x_2\omega(H)$. Indeed, if $x_1\omega(H) \cap x_2\omega(H) \neq \emptyset$, then $x_1HL \cap x_2HL \neq \emptyset$. Since $HL = LH$ is a subgroup of the group G (this follows from the normality of the group H), it follows that $x_1HL \equiv x_2HL$, i.e. $\omega(x_1H) = \omega(x_2H)$.

Denote by X the quotient set corresponding to the partition

$$\bigcup_{x \in G} \omega(xH),$$

and the quotient mapping $\omega(x) \mapsto \omega(xH)$ by

$$\lambda : \quad G/L \to X.$$

For brevity put $\tilde{x} = \lambda\omega(x)$, and define on X the metric

$$d(\tilde{x}_1, \tilde{x}_2) = \inf\{\rho(h_1\omega(x_1), h_2\omega(x_2)) : h_i \in H\} = \inf\{\rho(\omega(x_1), h_1^{-1}h_2\omega(x_2)) : h_i \in H\} = \inf\{\rho(\omega(x_1), h\omega(x_2)) : h$$

Lemma 3. *The mapping $\lambda : (G/L, \rho) \to (X, d)$ is open.*

Proof of the lemma. Let O be an open set in $(G/L, \rho)$, and let $\lambda^{-1}\lambda(O) = O$. For $\omega(x) \in O$ the number

$$\varepsilon = \inf\{\rho(\omega(x), \omega(\bar{x})) : \omega(\bar{x}) \in G/L \setminus O\} > 0.$$

If $d(\tilde{x}, \tilde{x}') < \varepsilon$, then there exists $h' \in H$ such that

$$\rho(\omega(x), h'\omega(x')) < \varepsilon.$$

Hence $h'\omega(x') \in O$, i.e. $\omega(x') \in O$. In other words, $\tilde{x}' \in \lambda(O)$. The openness of the mapping

$$\lambda : (G/L, \rho) \to (X, d)$$

now follows from the fact that the set

$$\lambda^{-1}\lambda(W) = HW = \bigcup_{h \in H} hW$$

is open whenever W is open. The lemma is proved.

As Michael proved ([4]), Theorem 1), there exists a set $A' \subset G/L$ such that $\lambda(A') = X$ and the mapping $\lambda|A' : A' \to X$ is perfect. In what follows we shall see that the set $A = \omega^{-1}(A')$ satisfies all the conditions of the theorem.

Consider the group G/H. The subgroup $\pi(L)$ decomposes G/H into left cosets. Assigning to each element $y \in G/H$ the coset $y\pi(L)$, we obtain the quotient mapping

$$\mu: \quad G/H \to Y,$$

where Y is the set of left cosets endowed with the quotient topology. Consider the diagram

$$\begin{array}{ccccc} G & \stackrel{\omega}{\longrightarrow} & G/L & \to & X \\ \downarrow \pi & & \downarrow \lambda & & \uparrow v \\ G/H & \stackrel{\mu}{\longrightarrow} & Y & & \end{array}$$

Since the mapping $\mu\pi$ is quotient, there exists a continuous mapping

$$v: \quad Y \to X$$

such that

$$\lambda\omega = v\mu\pi.$$

Lemma 4. *If an ultrafilter ξ in G is such that $\pi(\xi)$ converges to $\pi(x_1)$ and $\omega(\xi)$ converges to $\omega(x_2)$, then ξ converges to some $x_0 \in G$, with*

$$\pi(x_0) = \pi(x_1) \quad \text{and} \quad \omega(x_0) = \omega(x_2).$$

Proof of the lemma.

$$\lambda\omega(x_2) = \lambda \lim \omega(\xi) = \lim \lambda\omega(\xi) = \lim v\mu\pi(\xi) = v\mu \lim \pi(\xi) = \lambda\omega(x_1),$$

i.e. $x_1 \in x_2 HL$. Hence there exist $l \in L$ and $h \in H$ such that

$$x_1 h = x_2 l.$$

Put $x = x_1 h = x_2 l$. Then

$$x_1 H \cap x_2 L = xH \cap xL = x(H \cap L) = xK.$$

We shall prove that, for every neighborhood U of the identity of G, the set $xKU \in \xi$. To this end take a neighborhood $V \ni e$ such that

$$V^2 \subset U.$$

There exists a set $U_i \in \{U_n\}_{n=1}^{\infty}$ for which

$$U_i \cap H \subset KV.$$

Put

$$O = U_{i+1} \cap V.$$

The filter $\omega(\xi)$ converges to $\omega(x_2) =$

$= \omega(x)$, and therefore there is a set $B_2 \in \xi$ whose image $\omega(B_2) \subset \omega(xU_{i+2})$. Similarly, there exists $B_1 \in \xi$ such that $\pi(B_1) \subset \pi(xO)$. Then $B_1 \cap B_2 \in \xi$, and, moreover,

$$\begin{gathered} B_1 \cap B_2 \subset xOH \cap xU_{i+2}L = x(OH \cap U_{i+2}L) \subset \\ \subset x(OH \cap U_{i+2}^2) \subset x(HO \cap U_{i+1}) \subset \\ \subset x\{[(KV \cup (H \setminus KV))O] \cap U_{i+1}\} \subset \\ \subset x[(KVO \cup (H \setminus KV)O) \cap U_{i+1}] \subset \\ \subset x\{KV^2 \cup [(H \setminus KV)U_{i+1} \cap U_{i+1}]\} \subset xKV^2 \subset xKU \in \xi. \end{gathered}$$

Since ξ is an ultrafilter, it has at least one point of adherence $x_0 \in xK$. Then ξ converges to x_0. The lemma is proved.

Let us return to the set $A = \omega^{-1}(A')$. If for $x \in G$ we have $xH \cap A = xH \cap \omega^{-1}(A') = \emptyset$, then $\omega(xH) \cap A' = \emptyset$, which is impossible, since $\lambda(A') = X$. This proves that $\pi(A) = G/H$. The perfectness of the mapping $\pi/A : A \to G/H$ will follow from the following criterion ([3], p. 156, Theorem 1): a continuous mapping $\varphi : P \to Q$ of a separable topological space P onto a separable topological space Q is perfect if and only if every ultrafilter ξ in P whose image $\varphi(\xi)$ converges in Q converges in P. Let ξ be an ultrafilter in A for which $\lim \pi(\xi) = \pi(x)$. Then $\omega(\xi)$ is an ultrafilter in A', and moreover

$$\lim \lambda\omega(\xi) = \lim v\mu\pi(\xi) = v\mu \lim \pi(\xi) = v\mu\pi(x).$$

Since the mapping $\lambda/A' : A' \to X$ is perfect, $\omega(\xi)$ converges in A'. Let $\lim \omega(\xi) = \omega(\bar{x}) \in A'$, and let $\hat{\xi}$ be some ultrafilter in G containing the system of sets ξ. The filter $\pi(\hat{\xi}) = \pi(\xi)$ converges in G/H to $\pi(x)$, while $\omega(\hat{\xi})$ converges in G/L to $\omega(\bar{x})$. From Lemma 4 it follows that the filter $\hat{\xi}$ converges in G to some $x_0 \in G$, with $\pi(x_0) = \pi(x)$ and $\omega(x_0) = \omega(\bar{x})$, i.e. $x_0 \in \omega^{-1}\omega(x_0) \subset A$. Then the filter ξ converges in A to x_0. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.