ON A PROBLEM OF A. V. BITSADZE

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Abstract

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Let Ω be a bounded simply connected domain of the three-dimensional Euclidean space of points (x, y, z), bounded by a piecewise smooth surface σ, situated in the half-space $z > 0$, and by two surfaces S_1: $x + x_0 = \sqrt{y^2 + z^2}$ and S_2: $x_0 - x = \sqrt{y^2 + z^2}$, $x_0 > 0$, lying in the half-space $z < 0$; $\Omega_1 = \Omega \cap (z < 0)$, $\Omega_2 = \Omega \cap (z > 0)$; S_0 is the part of the plane $z = 0$ separating Ω_1 from Ω_2; $\partial\Omega$ is the boundary of Ω.

In the domain Ω consider the three-dimensional Lavrent' ev–Bitsadze equation

$$Lu \equiv \operatorname{sign} z \cdot u_{xx} + u_{yy} + u_{zz} = f(x, y, z), \tag{1}$$

which is elliptic for $z > 0$, hyperbolic for $z < 0$, and parabolically degenerates for $z = 0$. The surfaces S_1 and S_2 are characteristics of this equation.

Problem of A. V. Bitsadze. Find a function $u(x, y, z)$, continuous in the closed domain $\overline{\Omega}$, with first-order derivatives continuous inside Ω, satisfying equation (1) in the domain Ω for $z \neq 0$ and the boundary condition

$$u|_{\sigma} = 0, \qquad u|_{S_1} = 0 \tag{2}$$

or

$$u|_{\sigma} = 0, \qquad u|_{S_2} = 0. \tag{2*}$$

We shall adopt the following notation: $W_2^k(\Omega)$ is the Sobolev space with norm $\|\cdot\|_k$ and scalar product $\langle\cdot,\cdot\rangle_k$, $k = 0, 1$; $W(W^*)$ is the set of all functions u of the class $C(\overline{\Omega}) \cap C^2(\Omega \setminus S_0) \cap W_2^1(\Omega) \cap W_2^1(\partial\Omega)$, for which $Lu \in W_2^0(\Omega) = L_2(\Omega)$ and condition (2) ((2*)) is satisfied; $n = (x_n, y_n, z_n)$ is the unit outward normal to $\partial\Omega$, and $n^* = (-x_n, y_n, z_n)$ is the conormal corresponding to the Lorentz operator $\Box \equiv -\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$; $lu \equiv au + bu_x + cu_y + du_z$ is a first-order differential expression with coefficients $a(x, y, z) \in C^1(\overline{\Omega}) \cap C^2(\overline{\Omega}_1) \cap C^2(\overline{\Omega}_2)$, $b(x, y, z), c(x, y, z), d(x, y, z) \in C(\overline{\Omega}) \cap C^1(\overline{\Omega}_1) \cap C^1(\overline{\Omega}_2)$; A is the set of all four-component vector functions (a, b, c, d) which, in the domain Ω for $z \neq 0$, satisfy the system of differential inequalities

$$La > 0, \qquad \operatorname{sign} z \cdot b_y + c_x = 0, \qquad \operatorname{sign} z \cdot b_z + d_x = 0, \qquad c_z + d_y = 0; \quad (3)$$

$$b_x - c_y + d_z > 2a, \qquad b_x + c_y - d_z > 2a, \qquad d(x, y, z) = 0; \quad (4)$$

$$\operatorname{sign} z \cdot (c_y + d_z - b_x - 2a) > 0, \qquad \forall (x, y, z) \in \overline{\Omega}, \quad (5)$$

and the boundary conditions

$$a = 0, \qquad d = 0, \qquad bx_n \pm cy_n \leq 0, \qquad b^2 \geq c^2, \qquad \forall (x, y, z) \in S_2. \quad (6)$$

The set A is nonempty. It contains, for example, vectors whose components have the form

$$a = \varepsilon\,[y^2 + z^2 - (x - x_0)^2], \qquad b = b_1(x - x_0) + b_0, \qquad c = c_1 y, \qquad d = d(z), \quad (7)$$

where $\varepsilon, b_1, b_0, c_1$ are arbitrary constants such that

$$\varepsilon > 0, \qquad b_0 \leqslant 0, \qquad c_1 > |2a|, \qquad b_1 > c_1 + |2a|, \qquad \forall (x, y, z) \in \overline{\Omega}, \quad (8)$$

and the function $d(z) = 0$ for $z \leqslant 0$ and satisfies the differential inequality

$$2a + b_1 - c_1 < d'(z) < -2a + b_1 + c_1 \quad (9)$$

for $z \geqslant 0$. For $c_1 = 1,\ b_1 = \lambda,\ 1 < \lambda < 2$, and sufficiently small positive values of ε, the function $d = z$ is a solution of relation (9).

The inequalities (6) follow directly from (7) and (8), if one takes into account that on the characteristic surface S_2 we have $\sqrt{2x_n} = 1$, $\sqrt{2}(x_0 - x)y_n = y$, $\sqrt{2}(x_0 - x)z_n = z$. The validity of relations (3), (4), and (5) is obvious under the conditions (8) and (9).

Below we shall assume that the piecewise smooth surface σ has the property that, at least for one vector $(a, b, c, d) \in A$, almost everywhere on σ the inequality

$$n \cdot (b, c, d) = bx_n + cy_n + dz_n \geqslant 0 \quad (10)$$

holds.

A priori estimate. *For any function* $u \in W$ *the energy inequality*

$$\|u\|_1 \leqslant C\|Lu\|_0, \tag{11}$$

holds, where C *is a constant independent of* u.

Indeed, for any function $u \in W$ and any vector $(a, b, c, d) \in A$, the identity

$$\begin{aligned}
2\langle lu, Lu\rangle_0 &= 2\langle lu, \Delta u\rangle_0 + 2\langle lu, \Box u\rangle_0 \\
&= 2\int_{S_2} u^2 \frac{\partial a}{\partial n^*}\, dS = \int_\Omega u^2 La\, d\Omega + \int_\sigma [(bx_n - cy_n - dz_n)u_x^2 \\
&\quad + (-bx_n + cy_n - dz_n)u_y^2 + (-bx_n - cy_n + dz_n)u_z^2 + 2(cx_n + by_n)u_{xu}y \\
&\quad + 2(dx_n + bz_n)u_{xu}z + 2(dy_n + cz_n)u_{yu}z]\, dS \\
&\quad + \int_{S_1\cup S_2} [(-bx_n + cy_n + dz_n)u_x^2 + (-bx_n + cy_n - dz_n)u_y^2 \\
&\quad + (-bx_n - cy_n + dz_n)u_z^2 + 2(-cx_n + by_n)u_{xu}y \\
&\quad + 2(-dx_n + bz_n)u_{xu}z + 2(dy_n + cz_n)u_{yu}z]\, dS \\
&\quad + \int_\Omega [\operatorname{sign} z \cdot (c_y + d_z - b_x - 2a)u_x^2 + (b_x - c_y + d_z - 2a)u_y^2 \\
&\quad + (b_x + c_y - d_z - 2a)u_z^2]\, d\Omega = I_1 + I_2 + I_3 + I_4 + I_5, \qquad I_3 = I_3(S_1) + I_3(S_2).
\end{aligned}$$

Since $a = 0$ on S_2, and, as is well known, the conormal n^* lies on the characteristic S_2, it is obvious that $I_1 = 0$. The surface σ is a level surface; consequently, on it $u_x = u_{nx}n$, $u_y = u_{ny}n$, $u_z = u_{nz}n$. Taking this into account, it is easy to see that

$$I_3 = \int_\sigma (bx_n + cy_n + dz_n)|n|^2 u_n^2\, dS = \int_\sigma (b, c, d)\, n u_n^2\, dS,$$

from which, on the basis of (10), we conclude that $I_3 \geqslant 0$.

In a completely analogous way we obtain

$$I_3(S_1) = \int_\sigma (b, c, d)\, n\, (y_n^2 + z_n^2 - x_n^2)\, u_n^2\, dS = 0.$$

The matrix M of the quadratic form under the sign of the integral $I_3(S_2)$,

by virtue of (5) has the form

$$M = \left\| \begin{array}{ccc} cy_n - ax_n & by_n - cx_n & bz_n \\ by_n - cx_n & cy_n - bx_n & cz_n \\ bz_n & cz_n - cy_n & -bx_n \end{array} \right\|.$$

Relying exclusively on the fact that on the characteristic S_2, $y_n^2 + z_n^2 = x_n^2$, it is not difficult to show that, when inequality (6) is satisfied, all the principal minors of the matrix M are nonnegative: $\det M = 0$, $(cy_n - bx_n)^2 - (by_n - cx_n)^2 = (b^2 - c^2)z_n^2$, $-(c^2y_n^2 - b^2x_n^2) - c^2z_n^2 = (b^2 - c^2)x_n^2$, $-(c^2y_n^2 - b^2x_n^2) - b^2z_n^2 = (b^2 - c^2)y_n^2$.

Consequently, by Sylvester' s criterion, $I_3(S_2) \geqslant 0$. Thus,

$$\int_\sigma \big[u^2 La + \operatorname{sign} z \cdot (c_y + d_z - b_x - 2a)u_x^2 + (b_x - c_y + d_z - 2a)u_y^2 +$$

$$+ (b_x + c_y - d_z - 2a)u_z^2\big]\, d\Omega \leqslant \varepsilon \|u\|_1^2 + C_1 \|Lu\|_0^2, \tag{12}$$

where ε is an arbitrarily small positive number, and C_1 is a positive constant independent of u.

From the energy inequality (12) and conditions (3), (4), (5), (11) follows.

In exactly the same way one proves the validity of the a priori estimate (11) for any function $u \in W^*$.

Inequality (11) generalizes the a priori estimate obtained by A. V. Bitsadze in paper ([1]).

By a simple integration by parts one can show that

$$\langle u, Lv \rangle_0 = \langle v, Lu \rangle_0, \qquad \forall u \in W, \quad v \in W^*,$$

and therefore problems (1)–(2) and (1)–(2*) are (formally) mutually adjoint.

From the a priori estimate (11) there follows the uniqueness of the regular (classical) solution $u \in W$ or $v \in W^*$ of A. V. Bitsadze' s problem.

For $\sigma = S^*$, where S^* consists of the two conical surfaces $S_3: \; x_0 - x = \sqrt{y^2 + z^2}$ and $S_4: \; x + x_0 = \sqrt{y^2 + z^2}$, this fact was first established in ([1]).

Let now the right-hand side $f(x, y, z)$ of equation (1) belong to the Hilbert space $W_2^{-1}(\Omega)$ with negative norm $\|\cdot\|_{-1}$ and with scalar product $\langle \cdot, \cdot \rangle_{-1}$.

A **weak solution** of A. V. Bitsadze' s problem will be any function $u \in L_2(\Omega)$ satisfying the equality

$$\langle u, Lv \rangle_0 = \langle f, v \rangle_{-1}, \qquad \forall v \in W^*.$$

The proof of existence of a weak solution is carried out according to the usual scheme (see, for example, ([2]), p. 152; ([3]), p. 107), which we reproduce for convenience of reading.

According to the a priori estimate (11), valid for any function $v \in W^*$, the expression $\langle f, v\rangle_{-1}$ depends not on v, but on Lv, and therefore one may set $\langle f, v\rangle_{-1} = F(Lv)$, where F is a homogeneous and additive functional on the linear set $L(W^*)$. Further, on the basis of (11) and the generalized Schwarz inequality, we have

$$|F(Lv)| = |\langle f, v\rangle_{-1}| \leqslant \|f\|_{-1}\|v\|_1 \leqslant C\|f\|_{-1}\|Lv\|_0,$$

i.e., for fixed $f \in W_2^{-1}(\Omega)$, the functional $F(\varphi)$, $\varphi = Lv$, on $L(W^*)$ is continuous. Extending $F(\varphi)$ by the well-known Hahn–Banach theorem to the whole space $L_2(\Omega)$ and using the Riesz theorem, we find the desired function u: $F(\varphi) = \langle \varphi, u\rangle_0$, $\forall \varphi \in L_2(\Omega)$, and, in particular, for $\varphi = Lv$, $\langle Lv, u\rangle_0 = F(Lv) = \langle f, v\rangle_{-1}$.

The energy inequality (11)

$$\|u\|_1 \leq C\|Lu\|_0, \qquad \forall u \in W \cup W^*$$

also ensures the existence of a semistrong solution of A. V. Bitsadze' s problem (see ([3]), pp. 98-99).

Among the works devoted to the study of boundary-value problems for an equation of mixed type in multidimensional bounded domains, one should note the work ([4]), where a problem of the Dirichlet-problem type is studied.

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Note: Figure translations are in progress. See original paper for figures.