ISOMETRICITY OF CLOSED MANIFOLDS OF CONSTANT NEGATIVE CURVATURE WITH THE SAME FUNDAMENTAL GROUP

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Abstract

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G. A. MARGULIS

ISOMETRICITY OF CLOSED MANIFOLDS OF CONSTANT NEGATIVE CURVATURE WITH THE SAME FUNDAMENTAL GROUP

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I. Let M_1^n and M_2^n be two compact manifolds of constant negative curvature -1. In Mostow's paper (^{1}) it is proved that if M_1^n and M_2^n ($n \geqslant 3$) are diffeomorphic, then they are isometric. In the proof of this theorem the apparatus of the theory of quasiconformal mappings is used in an essential way. In the present note we formulate and present a brief outline of a proof of a theorem generalizing Mostow's theorem.

Theorem. *If the fundamental groups of two compact manifolds M_1^n and M_2^n ($n \geqslant 3$) of constant negative curvature -1 are isomorphic as abstract groups, then M_1^n and M_2^n are isometric.*

II. Let Γ be an abstract group with a finite number of generators $a_1, \dots, a_n$. Then, if $\gamma \in \Gamma$, by $\rho(\gamma)$ we denote the least length of a word by which γ is written in terms of $a_1, \dots, a_n$. Next, set

$$\hat{\rho}(\gamma_1, \gamma_2) = \rho(\gamma_1^{-1}\gamma_2), \tag{1}$$

where $\gamma_1, \gamma_2 \in \Gamma$. It is easy to see that $\hat{\rho}$ defines on Γ a left-invariant metric, i.e.

$$\hat{\rho}(\gamma\gamma_1, \gamma\gamma_2) = \hat{\rho}(\gamma_1, \gamma_2) \tag{2}$$

for any $\gamma_1, \gamma_2, \gamma \in \Gamma$.

We shall say that two metrics ρ_1 and ρ_2 on a space X are equivalent if there exist constants c_1 and c_2 such that for any $x_1, x_2 \in X$

$$0 < c_1 < \frac{\rho_1(x_1, x_2)}{\rho_2(x_1, x_2)} < c_2 < \infty. \tag{3}$$

If $\tilde{a}_1 \ldots \tilde{a}_m$ is another finite system of generators of the group Γ, then for this system one can analogously define a metric $\hat{\hat{\rho}}$ corresponding to the metric $\hat{\rho}$. Then it is easily proved that the metrics $\hat{\rho}$ and $\hat{\hat{\rho}}$ are equivalent.

III. Denote by Γ the group to which both $\pi_1(M_1^n)$ and $\pi_1(M_2^n)$ are isomorphic, and by h_1 and h_2 the corresponding isomorphisms. Choose in L^n some fixed point x, and for any $\gamma_1, \gamma_2 \in \Gamma$ set

$$\bar{\rho}(\gamma_1, \gamma_2) = \rho_{L^n}[h_1(\gamma_1)(x),\, h_1(\gamma_2)(x)], \tag{4}$$

$$\tilde{\rho}(\gamma_1, \gamma_2) = \rho_{L^n}[h_2(\gamma_1)(x),\, h_2(\gamma_2)(x)], \tag{5}$$

where ρ_{L^n} is the distance in Lobachevsky space, and $h_1(\gamma_1)(x)$, $h_1(\gamma_2)(x)$, $h_2(\gamma_1)(x)$, $h_2(\gamma_2)(x)$ are the images of the point x under the action of the transformations $h_1(\gamma_1)$, $h_1(\gamma_2)$, $h_2(\gamma_1)$, $h_2(\gamma_2)$ (here the groups $\pi_1(M_1^n)$ and $\pi_1(M_2^n)$ are regarded as subgroups of the group of motions of Lobachevsky space).

Since $\pi_1(M_1^n)$ and $\pi_1(M_2^n)$ are the fundamental groups of compact manifolds, Γ is a group with a finite number of generators. Po-

therefore on Γ one can define a metric $\hat{\rho}$ by the method described in Sec. II. It can be shown that both the metric $\tilde{\rho}$ and the metric $\hat{\rho}$ are equivalent to the metric ρ. Therefore the metrics $\tilde{\rho}$ and $\hat{\rho}$ are equivalent to each other.

Denote by X_1 the set $\pi_1(M_1^n)(x)$, and by X_2 the set $\pi_1(M_2^n)(x)$. The sets X_1 and X_2 are subsets of Lobachevskii space. Let ρ_{X_1} and ρ_{X_2} be the restrictions of the metric of L^n to X_1 and X_2. Between X_1 and Γ, as well as between X_2 and Γ, a natural one-to-one correspondence is established (namely, if $\gamma \in \Gamma$, then set $g_1(\gamma) = h_1(\gamma)(x)$ and $g_2(\gamma) = h_2(\gamma)(x)$). Consider the mapping $X_1 \xrightarrow{g} X_2$ defined by the formula

$$g = g_2 g_1^{-1}. \tag{6}$$

From the fact that the metrics $\tilde{\rho}$ and $\hat{\rho}$ are equivalent, it follows easily that the mapping g satisfies the Lipschitz condition (with respect to the metrics ρ_{X_1} and ρ_{X_2}). Moreover, the sets X_1 and X_2 are sufficiently dense in L^n, i.e., there exists a constant $D > 0$ such that every ball of radius D in L^n contains both a point of X_1 and a point of X_2. On this basis one can prove that the mapping g extends continuously to the absolute of the space L^n (see ([2])), and the restriction of this mapping to the absolute is quasiconformal. After this, using the method of work ([1]), the theorem formulated at the beginning is proved.

Institute for Problems of Information Transmission
Academy of Sciences of the USSR
Moscow

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Note: Figure translations are in progress. See original paper for figures.

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