

**SPECTRUM OF A
SYSTEM OF
MEMBRANE
EQUATIONS IN THE
CASE OF
AXISYMMETRIC
VIBRATIONS OF A
SHELL OF
REVOLUTION**

THEORY OF ELASTICITY

1970

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-197001.20260>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 539.3

THEORY OF ELASTICITY

V. B. LIDSKII, N. V. KHARKOVA

SPECTRUM OF A SYSTEM OF MEMBRANE EQUATIONS IN THE CASE OF AXISYMMETRIC VIBRATIONS OF A SHELL OF REVOLUTION

(Presented by Academician A. Yu. Ishlinskii, 18 III 1970)

1. The system of equations for axisymmetric vibrations of a shell of revolution has the form

$$-\frac{d}{ds} \left(\frac{1}{B(s)} \frac{dB(s)u}{ds} \right) - \frac{1-\sigma}{R_1(s)R_2(s)} u + \left(\frac{1}{R_1} + \frac{\sigma}{R_2} \right) \frac{dw}{ds} + \frac{d}{ds} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) w = \lambda u, \quad (1)$$

$$\frac{h^2}{12} \frac{1}{B} \frac{d}{ds} \left(B \frac{d}{ds} \left(\frac{1}{B} \frac{d}{ds} \left(B \frac{dw}{ds} \right) \right) \right) - \left(\frac{1}{R_1} + \frac{\sigma}{R_2} \right) \frac{du}{ds} - \left(\frac{\sigma}{R_1} + \frac{1}{R_2} \right) \frac{B'_s}{B} u + \left(\frac{1}{R_1^2} + \frac{2\sigma}{R_1 R_2} + \frac{1}{R_2^2} \right) w = \lambda w. \quad (1')$$

Here s is the length of the meridian arc, $a \leq s \leq b$; $B(s)$ is the abscissa of the meridian of the middle surface, and $R_1^{-1}(s)$ and $R_2^{-1}(s)$ are the principal curvatures of the surface of revolution:

$$R_1^{-1}(s) = -B''_{ss}(1 - B_s'^2)^{-1/2}; \quad R_2^{-1}(s) = (1 - B_s'^2)^{1/2} B^{-1}; \quad (2)$$

σ is Poisson's ratio; λ is the spectral parameter, equal to $(1 - \sigma^2)p^2 \rho E^{-1}$, where ρ is the density, E is Young's modulus, and p is the vibration frequency.

The functions $u(s)$ and $w(s)$ are the components of the displacement vector along the meridian and along the normal to the surface; h is a small parameter.

In the case of rigid clamping of the shell edges, the boundary conditions have the form

$$u(a) = u(b) = w(a) = w(b) = w'_s(a) = w'_s(b) = 0 \quad (3)$$

(see, in this connection, ^(1,2,5)).

To find the zero approximations of the frequencies of problem (1), (3) with respect to the small parameter h , one sets $h = 0$ in (1'). The resulting system of differential equations of second order will be written briefly in the form

$$L_0 f = \lambda f, \quad (4)$$

where $f(s) = (u(s), w(s))$ is a vector function, and we shall study the spectrum of system (4) under the boundary conditions

$$u(a) = u(b) = 0. \quad (4')$$

It can be shown that the operator L_0 , in the space of smooth vector functions $f(s)$ satisfying the boundary conditions (4'), is symmetric and positive definite. It is assumed that the scalar product is introduced by the formula

$$(f_1, f_2) = \int_a^b B(s)(u_1(s)\bar{u}_2(s) + w_1(s)\bar{w}_2(s)) ds. \quad (5)$$

The closure of this operator* will be denoted by L_0 .

2. We rewrite system (3) in understandable notation

$$L_{11}u + L_{12}w = \lambda u, \quad (6)$$

$$L_{21}u + L_{22}w = \lambda w \quad (6')$$

and introduce the functions

$$\varphi_1(s) = \frac{1 - \sigma^2}{R_2^2(s)}; \quad \varphi_2(s) = \frac{1}{R_1^2(s)} + \frac{2\sigma}{R_1(s)R_2(s)} + \frac{1}{R_2^2(s)}. \quad (7)$$

It is not difficult to verify that $\varphi_2(s) \geq \varphi_1(s)$ ($s \in [a, b]$). Let us also denote the set of values of the function $\varphi_1(s)$ by $[\alpha, \beta]$, and the set of values of $\varphi_2(s)$ by $[\gamma, \delta]$. In what follows we shall assume that the function $B(s)$ (the abscissa of the meridian) is sufficiently smooth and positive.

We shall need the following auxiliary propositions.

Lemma 1. Let $s_0 \in [a, b]$, $\lambda \notin [\alpha, \beta]$, and $\lambda \neq \varphi_2(s_0)$. Let r_1 and r_2 be arbitrary real numbers. Then there exists a unique solution of system (4) $f(s, \lambda) = (u(s, \lambda), w(s, \lambda))$ satisfying the Cauchy conditions

$$u(s_0, \lambda) = r_1, \quad u'_s(s_0, \lambda) = r_2. \quad (8)$$

The vector function $f(s, \lambda)$, for any $s \in [a, b]$, is regular in the whole λ -plane with the segment $[\alpha, \beta]$ and the point $\lambda = \varphi_2(s_0)$, at which it has a simple pole, removed.

Lemma 1 is proved by reducing the Cauchy problem to the Volterra equation

$$(\varphi_1(s) - \lambda)w(s, \lambda) + \int_{s_0}^s K(s, t, \lambda)w(t, \lambda) dt = \tau(s, \lambda), \quad (9)$$

in which $K(s, t, \lambda)$ is an entire function of λ , while $\tau(s, \lambda)$ has a pole at $\lambda = \varphi_2(s_0)$.

Lemma 2. On the resolvent set of the operator L_{11} ** the spectrum of problem (4), (4') coincides with the spectrum of the equation

$$(\varphi_1(s) - \lambda)w + T(\lambda)w = T_1(\lambda)h_1 + h_2, \quad (10)$$

where $T(\lambda)$ and $T_1(\lambda)$ are meromorphic completely continuous operators with poles at the points of the spectrum of the operator L_{11} ; $h_1(s)$ and $h_2(s)$ are arbitrary functions from $\mathcal{L}_2(a, b)$.

Using Lemmas 1 and 2, the following is proved.

Theorem 1. The spectrum of problem (4), (4') for $\lambda \notin [\alpha, \beta]$ is discrete (consists of isolated eigenvalues of finite multiplicity). The eigenvalues have an accumulation point $\lambda = +\infty$ and, possibly, the points $\lambda = \alpha$ and $\lambda = \beta$. The entire segment $[\alpha, \beta]$ belongs to the spectrum of the operator L_0 . All eigenvalues not belonging to the segments $[\alpha, \beta]$ and $[\gamma, \delta]$ are simple; eigenvalues belonging to the segment $[\gamma, \delta]$ are at most double.

2. We shall now show that, for the eigenfunctions $f_k(s) = (u_k(s), w_k(s))$ of problem (4), (4'), oscillation theorems hold.

The following proposition is fundamental.

Theorem 2. Let $\lambda \notin [\alpha, \beta]$, $\lambda \notin [\gamma, \delta]$. Let $f_0(s, \lambda) = (u_0(s, \lambda), w_0(s, \lambda))$ be the solution of system (4) satisfying the condition $u_0(a, \lambda) = 0$,

* It is easy to show that the operator L_0 , defined on the manifold of smooth vector functions satisfying conditions (4'), is essentially self-adjoint.

** The operator L_{11} is defined by the formula

$$L_{11}u = -\frac{d}{ds} \left(\frac{1}{B} \frac{dBu}{ds} \right) - \frac{1 - \sigma}{R_1 R_2} u$$

and acts on scalar functions $u(s)$ satisfying condition (4'). The spectrum of L_{11} , as is well known, is real and discrete. The resolvent set is the complement of the spectrum.

$u'_0(a, \lambda) = 1$. Let $s_0(\lambda)$ be a zero of the function $u_0(s, \lambda)$ belonging to the interval (a, b) . Then the formula

$$\frac{ds_0}{d\lambda} = -\frac{\lambda - \varphi_2(s_0)}{\lambda - \varphi_1(s_0)} \left(\frac{\partial u_0(s_0)}{\partial s} \right)^{-2} \left(\int_a^{s_0} B(u_0^2 + w_0^2) ds \right) B^{-1}(s_0), \quad (11)$$

is valid, and consequently, for $\lambda < \alpha$ and $\lambda > \delta$ all zeros of the function $u_0(s, \lambda)$ move to the left as λ increases, while for $\beta < \lambda < \gamma$ they move to the right.

We agree to call the eigenvalues belonging to the interval $(0, \alpha)$ the first series, those belonging to the interval (β, γ) the second series, those belonging to the segment $[\gamma, \delta]$ the third series, and, finally, the eigenvalues belonging to the interval $(\delta, +\infty)$ the fourth series. Note that, according to Theorem 1, the fourth series is infinite, the third is finite, and the first and second may be finite or infinite*. Within each series we renumber the eigenvalues in increasing order.

Let $n_1(\lambda)$ be the number of zeros of the function $u_0(s, \lambda)$ for $s \in (a, b)$, when $\lambda < \alpha$, and let

$$N_1 = \sup_{\lambda < \alpha} n_1(\lambda) \quad (N_1 \leq +\infty). \quad (12)$$

Theorem 2 allows us to make the following conclusion:

Corollary 1. *The first series consists of N_1 eigenvalues $\lambda_0^{(1)}, \lambda_1^{(1)}, \dots, \lambda_k^{(1)}, \dots$, and the component $u_0(s, \lambda_k^{(1)})$ corresponding to the eigenfunction has exactly k zeros inside the interval (a, b) .*

Analogous assertions are valid for the second and fourth series. In particular, the first component of the k -th eigenfunction of the fourth series has $k + M_3$ zeros on the interval (a, b) , where $M_3 = \inf_{\lambda > \delta} n_3(\lambda)$, and $n_3(\lambda)$ is the number of zeros of $u_0(s, \lambda)$ for $\lambda > \delta$. Note that the eigenfunctions of the third series may have any number of zeros.

Theorem 3. *Suppose that in equation (4) the function $1/R_1(s) + \sigma/R_2(s)$ (see (1)) preserves its sign on the segment $[a, b]$. Then between any two neighboring zeros of the component $u(s, \lambda_k^{(i)})$ of the eigenfunction $f_k^{(i)}(s)$ of the first, second, and fourth series there lies strictly one zero of the component $w(s, \lambda_k^{(i)})$, $i = 1, 2, 4$. Thus, the zeros of the components of the eigenfunctions $f_k^{(i)}(s)$ strictly interlace.*

4. We now give a sufficient condition under which the first series of frequencies (the lowest) is infinite.

Theorem 4. *Suppose that in a neighborhood of the point $s = s_0 \in [a, b]$ the function $B(s)$ attains a maximum, so that $B(s) = B_0 - \frac{1}{2}k(s - s_0)^2 + O[(s - s_0)^3]$. Suppose $0 \leq B_0 k \leq 1$, and the local minimum attained by $\varphi_1(s)$ at the point s_0 coincides with its infimum $\varphi_1(s_0) = \alpha > 0$. Then, under the condition*

$$9(kB_0)^2 + (12\sigma - 1)kB_0 + 4\sigma^2 > 0 \quad (13)$$

the first series is infinite. For $\sigma > 1/24$, all kB_0 satisfy this condition.

The proof is carried out by studying the zeros of the solution $u_0(s, \lambda)$ and is based on Corollary 1 of Theorem 2.

In conclusion, note that in the case of a spherical shell the first series is infinite (this fact has already been noted in the literature, see ^(3,4)), while in the case of a conical shell it is, generally speaking, not empty. The number of eigenvalues in the first series can be made arbitrarily large; for this it is sufficient to take a cone with the angle of inclination of the generator close to $\pi/2$.

* Of course, the first, second, and third series may be absent altogether.

The authors are grateful to A. L. Goldenveizer, on whose initiative this study was undertaken, as well as to E. P. Tovstik, G. N. Chernyshev, and D. I. Tselnik for discussions and advice.

Moscow Institute of Physics and Technology

Institute for Problems in Mechanics
Academy of Sciences of the USSR
Moscow

Received
16 III 1970

REFERENCES

1. A. L. Goldenveizer, *Theory of Elastic Thin Shells*, Moscow, 1953.
2. P. E. Tovstik, *Studies in Elasticity and Plasticity*, Leningrad Univ., collection no. 4, 1965.
3. P. E. Tovstik, *Izv. AN SSSR, Mekhanika*, no. 6, 111 (1965).
4. O. V. Luzhin, *Collection: Studies in the Theory of Structures*, vol. 10, 1961.
5. G. I. Pshenichnov, *Proceedings of the Sixth All-Union Conference on the Theory of Shells and Plates*, "Nauka," 1966, p. 64.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.