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Abstract

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PHYSICS

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CHRONOMETRICALLY INVARIANT REPRESENTATION OF PETROV' S CLASSIFICATION OF GRAVITATIONAL FIELDS

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The purpose of the present work is a chronometrically invariant formulation of Petrov' s algebraic classification of gravitational fields. The expression of the algebraic characteristics of gravitational fields in terms of chronometric invariants makes it possible in a number of cases to give them a physical interpretation in a given reference system, and also to obtain chronometrically invariant conditions characterizing gravitational fields of different types. The consideration will be carried out in an orthoframe chosen at a fixed point of the space-time V_4 of general relativity.

Four-dimensional coordinate systems belonging to one and the same reference system are, by definition, related to one another by transformations of the form ([1])

$$\begin{aligned} \tilde{x}^0 &= \tilde{x}^0(x^0, x^1, x^2, x^3), \quad \text{(a)} \\ \tilde{x}^i &= \tilde{x}^i(x^1, x^2, x^3), \quad \text{(b)} \\ \partial \tilde{x}^i / \partial x^0 &= 0. \end{aligned} \tag{1}$$

We shall agree that Greek indices take the values $0, 1, 2, 3$, Latin indices $1, 2, 3$; the spatial coordinates will be denoted by x^1, x^2, x^3, and the time coordinate by x^0 ($x^0 = ct$, where c is the fundamental speed). In a chosen reference system, three-dimensional tensor quantities invariant with respect to (1a), i.e. chronometrically invariant (c.i.) quantities, are physically distinguished. The three-dimensional space of a given reference system is characterized ([1]) by three mechanical c.i. quantities: F^i (the vector of the gravitational-inertial force), A_{ik} (the tensor of the angular velocity of rotation relative to a locally comoving geodesic coordinate system), D_{ik} (the tensor of deformation velocities), and one geometrical quantity (the curvature tensor of the three-dimensional space). In an arbitrary fixed reference system, the 20 essential components of the Weyl conformal-curvature tensor $C_{\alpha\beta\gamma\delta}$ can be divided into 3 c.i. three-dimensional tensors (c.i. components of the Weyl tensor) ([2]):

$$\widetilde{X}^{ij} = -c^2 \frac{C_{0\cdot 0\cdot}^{i\cdot j}}{g_{00}}, \qquad \widetilde{Y}^{ijk} = -\frac{c\, C_{0\cdot\cdot}^{ijk}}{\sqrt{g_{00}}}, \qquad \widetilde{Z}^{iklj} = c^2 C^{iklj}, \tag{2}$$

possessing the following properties:

$$\widetilde{X}_{ij} = \widetilde{X}_{ji}, \qquad \widetilde{X}_k^k = 0, \qquad \widetilde{Y}_{[ijk]} = 0, \qquad \widetilde{Y}_{ijk} = -\widetilde{Y}_{ikj}. \tag{3}$$

Here $\widetilde{X}_k^i = h^{ik} X_{ik}$, where h_{ik} is the c.i. three-dimensional metric tensor ($h_{ik} = -g_{ik} - g_{0i}g_{0k}/g_{00}$). The tensor $\widetilde{Z}_{iklj}$ possesses all the algebraic properties of the curvature tensor.

In an orthoframe, equalities (2) take the form

$$\widetilde{X}_{ij} = -c^2 C_{0i0j}, \qquad \widetilde{Y}_{ijk} = -cC_{0ijk}, \qquad \widetilde{Z}_{iklj} = c^2 C_{iklj}. \tag{4}$$

Expressing the components of the Weyl tensor through c.i. quantities, we obtain for them in the orthoframe

$$C_{i0j0} = -\frac{1}{c^2} X_{ij} - \frac{\varkappa}{2c^2} U_{ij} + \frac{\varkappa\rho}{6} h_{ij} + \frac{\varkappa c^2}{3} U h_{ij}, \tag{5}$$

$$C_{i0jk} = \frac{1}{c} Y_{ijk} - \frac{\varkappa}{2c}(h_{ik}J_j - h_{ij}J_k), \tag{6}$$

$$\begin{aligned} C_{iklj} = \frac{1}{c^2} Z_{iklj} - \frac{\varkappa}{2c^2}(h_{ij}U_{kl} - h_{il}U_{kj} + h_{kl}U_{ij} - h_{kj}U_{il}) - \\ -\frac{1}{3}\varkappa(\rho - U/c^2)(h_{ik}h_{jl} - h_{il}h_{jk}), \end{aligned} \tag{7}$$

where X_{ij}, Y_{ijk}, Z_{iklj} are the ch.i. components of the curvature tensor $R_{\alpha\beta\gamma\delta}$, defined analogously to (2) and expressed in terms of the ch.i. characteristics of the reference space (F_i, A_{ik}, D_{ik} and the three-dimensional curvature tensor) and their first derivatives (2); $\rho = T_{00}/g_{00}$ is the mass density; $J^i = cT_0^i/\sqrt{g_{00}}$ is the vector of the mass-flow density (momentum); $U^{ij} = c^2 T^{ij}$ is the tensor of the momentum-flow density (stress tensor) ([1]).

Writing the equations $C_{\alpha\beta} = 0$ in the orthoframe, taking into account relations (4), and introducing the three-dimensional matrices $\tilde{x}$ and $\tilde{y}$:

$$\tilde{x} \equiv \|\tilde{x}_{ik}\| \overset{\text{def}}{=} -\frac{1}{c^2}\|\widetilde{X}_{ik}\|, \qquad \tilde{y} \equiv \|\tilde{y}_{ik}\| \overset{\text{def}}{=} \frac{1}{2c}\|\varepsilon_{imn}\widetilde{Y}_{kmn}\| \tag{8}$$

(ε_{imn} is the three-dimensional discriminant tensor), one can represent the six-dimensional matrix $\|C_{ab}\|$ in the following form*

$$\|C_{ab}\| = \left\| \begin{matrix} \tilde{x} & \tilde{y} \\ \tilde{y} & -\tilde{x} \end{matrix} \right\|. \tag{9}$$

In this case the following relations hold:

$$\tilde{x}_{11} + \tilde{x}_{22} + \tilde{x}_{33} = 0, \qquad \tilde{y}_{11} + \tilde{y}_{22} + \tilde{y}_{33} = 0. \tag{10}$$

Let us form the λ-matrix

$$\|C_{ab} - \lambda g_{ab}\| = \left\| \begin{matrix} \tilde{x} + \lambda\varepsilon & \tilde{y} \\ \tilde{y} & -\tilde{x} - \lambda\varepsilon \end{matrix} \right\|,$$

where ε is the three-dimensional unit matrix, and, using elementary transformations, reduce it to the form:

$$\left\| \begin{matrix} \tilde{x} + i\tilde{y} + \lambda\varepsilon & 0 \\ 0 & \tilde{x} - i\tilde{y} + \lambda\varepsilon \end{matrix} \right\| \equiv \left\| \begin{matrix} Q(\lambda) & 0 \\ 0 & \overline{Q}(\lambda) \end{matrix} \right\|. \tag{11}$$

The original λ-matrix may have one of the following characteristics ([3]):

$$1.\ [111, \overline{111}]. \qquad 2.\ [21, \overline{21}]. \qquad 3.\ [3, \overline{3}]. \tag{12}$$

Using the canonical form, obtained by A. Z. Petrov ([3]), of the matrix $\|R_{ab}\|$ in a nonholonomic orthoframe for each of the three types $\widetilde{T}_i$ $(i = 1, 2, 3)$, we express the matrix $\|C_{ab}\|$ in terms of the components of the ch.i. tensors $\widetilde{X}_{ij}$ and $\widetilde{Y}_{ijk}$.

1. Type $\widetilde{T}_1$

$$\|C_{ab}\| = \left\| \begin{matrix} \tilde{x} & \tilde{y} \\ \tilde{y} & -\tilde{x} \end{matrix} \right\|, \qquad \tilde{x} = \left\| \begin{matrix} \tilde{x}_{11} & 0 & 0 \\ 0 & \tilde{x}_{22} & 0 \\ 0 & 0 & \tilde{x}_{33} \end{matrix} \right\|, \qquad \tilde{y} = \left\| \begin{matrix} \tilde{y}_{11} & 0 & 0 \\ 0 & \tilde{y}_{22} & 0 \\ 0 & 0 & \tilde{y}_{33} \end{matrix} \right\|, \tag{13}$$

where

$$\tilde{x}_{11} + \tilde{x}_{22} + \tilde{x}_{33} = 0, \qquad \tilde{y}_{11} + \tilde{y}_{22} + \tilde{y}_{33} = 0. \tag{14}$$

It can be shown that in the orthoframe the following conditions are satisfied:

$$\tilde{y}_{11} = \frac{1}{c}\widetilde{Y}_{123} = \frac{1}{c}Y_{123}, \quad \tilde{y}_{22} = \frac{1}{c}\widetilde{Y}_{231} = \frac{1}{c}Y_{231}, \quad \tilde{y}_{33} = \frac{1}{c}\widetilde{Y}_{312} = \frac{1}{c}Y_{312}. \quad (15)$$

Then the stationary curvatures $\tilde{\lambda}_s$ $(s = 1, 2, 3)$ for the space $\widetilde{T}_1$ have the form:

$$\begin{aligned} \tilde{\lambda}_1 &= -\frac{1}{c^2}X_{11} + \frac{i}{c}Y_{123} - \frac{\varkappa}{2c^2}U_{11} + \frac{\varkappa\rho}{6} + \frac{\varkappa U}{3c^2}, \\ \tilde{\lambda}_2 &= -\frac{1}{c^2}X_{22} + \frac{i}{c}Y_{231} - \frac{\varkappa}{2c^2}U_{22} + \frac{\varkappa\rho}{6} + \frac{\varkappa U}{3c^2}, \\ \tilde{\lambda}_3 &= -\frac{1}{c^2}X_{33} + \frac{i}{c}Y_{312} - \frac{\varkappa}{2c^2}U_{33} + \frac{\varkappa\rho}{6} + \frac{\varkappa U}{3c^2}. \end{aligned} \quad (16)$$

* The connection of the six-dimensional matrix $\|R_{ab}\|$ of the curvature tensor with ch.i. quantities was first indicated, in another form, by G. A. Sokolik, who proceeded from the theory of tensor representations of linear unimodular groups.

For spaces of type D $(\tilde{\lambda}_2 = \tilde{\lambda}_3)$ we have: $X_{22} - X_{33} = \frac{1}{2}\varkappa(U_{33} - U_{22})$, $Y_{231} = Y_{312}$; for spaces of type O $(\tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}_3)$: $X_{11} + \frac{1}{2}\varkappa U_{11} = X_{22} + \frac{1}{2}\varkappa U_{22} = X_{33} + \frac{1}{2}\varkappa U_{33}$, $Y_{123} = Y_{231} = Y_{312} = 0$.

2. Type $\widetilde{T}_2$

$$\|C_{ab}\| = \begin{Vmatrix} \tilde{x} & \tilde{y} \\ \tilde{y} & -\tilde{x} \end{Vmatrix}, \qquad \tilde{x} = \begin{Vmatrix} \tilde{x}_{11} & 0 & 0 \\ 0 & \tilde{x}_{22} & 0 \\ 0 & 0 & \tilde{x}_{33} \end{Vmatrix}, \qquad \tilde{y} = \begin{Vmatrix} \tilde{y}_{11} & 0 & 0 \\ 0 & \tilde{y}_{22} & 1 \\ 0 & 1 & \tilde{y}_{22} \end{Vmatrix}; \quad (17)$$

where

$$\tilde{x}_{11} + \tilde{x}_{22} + \tilde{x}_{33} = 0, \qquad \tilde{y}_{11} + 2\tilde{y}_{22} = 0, \qquad \tilde{x}_{22} - \tilde{x}_{33} = 2; \quad (18)$$

$$\tilde{\lambda}_1 = -\frac{1}{c^2}X_{11} + \frac{i}{c}Y_{123} - \frac{\varkappa}{2c^2}U_{11} + \frac{\varkappa\rho}{6} + \frac{\varkappa U}{3c^2},$$

$$\tilde{\lambda}_2 = -\frac{1}{c^2}X_{22} - 1 + \frac{i}{c}Y_{231} - \frac{\varkappa}{2c^2}U_{22} + \frac{\varkappa\rho}{6} + \frac{\varkappa U}{3c^2} = -\frac{1}{c^2}X_{33} + 1 +$$

$$+ \frac{i}{c}Y_{231} - \frac{\varkappa}{2c^2}U_{22} + \frac{\varkappa\rho}{6} + \frac{\varkappa U}{3c^2}. \quad (19)$$

For spaces of type N ($\tilde{\lambda}_1 = \tilde{\lambda}_2$), the following relations hold in an orthoreper:

$$X_{22} - c^2 + \varkappa U_{22}/2 = X_{33} + c^2 + \varkappa U_{22}/2 = X_{11} + \varkappa U_{11}/2,$$

$$Y_{123} = Y_{231} = Y_{312} = 0.$$

3. Type $\tilde{T}_3$.

$$\|C_{ab}\| = \begin{Vmatrix} \tilde{x} & \tilde{y} \\ \tilde{y} & -\tilde{x} \end{Vmatrix}, \qquad \tilde{x} = \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \qquad \tilde{y} = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{Vmatrix}. \quad (20)$$

From formulas (13), (17), and (20) it follows that the condition $Y_{ijk} = 0$ can be satisfied only for fields of type $\tilde{T}_1$. Hence one may conclude that gravitational fields for which, in some reference frame, the condition $Y_{ijk} = 0$ is fulfilled can belong only to type $\tilde{T}_1$, with material stationary curvatures.

Let us consider an example of metrics of type $\tilde{T}_1$ satisfying the condition $Y_{ijk} = 0$. From the expression for Y_{ijk} (2)

$$Y_{ijk} = {}^*\nabla_j(D_{ik} + A_{ik}) - {}^*\nabla_i(D_{jk} + A_{jk}) + \frac{2}{c^2}A_{ji}F_k - \frac{\varkappa}{2}(h_{ik}J_j - h_{ij}J_k) \quad (21)$$

and the Einstein equations in ch.i. form (1), it follows that the ch.i. tensor Y_{ijk} vanishes in reference frames in which any one of the following conditions is satisfied:

$$^*\nabla_{jD_{ik}} = 0, \qquad ^*\nabla_{jA_{ik}} = 0, \qquad F_i = 0; \quad (22)$$

$$^*\nabla_{jD_{ik}} = 0, \qquad A_{ik} = 0, \quad (23)$$

where $^*\nabla_i$ is the symbol of ch.i. spatially covariant differentiation. If the stronger condition $A_{ik} = 0$, $D_{ik} = 0$ is satisfied, then we arrive at a reference frame that is accelerated, without rotating and without being deformed in the process*. The condition $A_{ik} = 0$ makes it possible, in this reference frame, to choose the time coordinate x^0 so that the condition $g_{0i} = 0$ holds [1]. The condition $D_{ik} = 0$ leads to the stationarity of the ch.i. three-dimensional tensor h_{ik} [1]. Since in this reference frame $h_{ik} = -g_{ik}$, the three-dimensional metric g_{ik} can be transformed to diagonal form. Thus the metric ds^2 in this reference frame can be written in the form

$$ds^2 = g_{00}(x^0, x^1, x^2, x^3)\, dx^{0^2} + g_{11} dx^{1^2} + g_{22} dx^{2^2} + g_{33} dx^{3^2}, \tag{24}$$

where $g_{ii} = g_{ii}(x^1, x^2, x^3)$.

* Here, in speaking of accelerated motion, rotation, and deformation of the reference frame, we mean the corresponding motions of the three-dimensional space of this reference frame relative to the locally accompanying geodesic coordinate system.

Let us consider two special cases.

1. $g_{00} = g_{00}(x^0)$, then $F_i = 0$ ([5]), i.e., the reference frame is freely falling. It can be shown that in this case the Einstein space with metric (24) is flat space-time ([3]). Thus, in an Einstein space ($R_{\alpha\beta} = \varkappa g_{\alpha\beta}$) the three chronometrically invariant conditions $F_i = 0,\ A_{ik} = 0,\ D_{ik} = 0$ cannot hold simultaneously, except in the case of flat space-time.
2. $g_{00} = g_{00}(x^1, x^2, x^3)$, i.e., ${}^*\partial F_i/\partial t = 0$ ([5]) (a stationary force field). In this case the metric (24) is the general form of the metric of a static space $[g_{\alpha\beta} = g_{\alpha\beta}(x^1, x^2, x^3),\ g_{0i} = 0]$ ([3]). In other words, every static space-time V_4 admits a reference frame in which the conditions ${}^*\partial F_i/\partial t = 0,\ A_{ik} = 0,\ D_{ik} = 0$ are satisfied.

From consideration of the canonical forms (13), (17), and (20), the definition (8) of the matrix $\|\tilde{y}_{ik}\|$, and condition (15), it follows that V_4 has real stationary curvatures if and only if, in the chosen orthonormal frame of the given point, the condition is satisfied:

$$Y_{123} = Y_{231} = Y_{312} = 0. \tag{25}$$

In vacuum the Weyl tensor coincides with the curvature tensor $R_{\alpha\beta\gamma\delta}$, and we can classify empty spaces using the λ-matrix

$$\|R_{ab} - \lambda g_{ab}\| = \left\| \begin{matrix} x + \lambda\varepsilon & y \\ y & -x - \lambda\varepsilon \end{matrix} \right\|, \tag{26}$$

where

$$x = \|x_{ik}\| \stackrel{\text{def}}{=} -\frac{1}{c^2}\|X_{ik}\|, \qquad y = \|y_{ik}\| \stackrel{\text{def}}{=} \frac{1}{2c}\|e_{imn}\tilde{Y}_{kmn}\|. \tag{27}$$

Reducing the λ-matrix by elementary transformations to the quasidiagonal form (11), we obtain the three Petrov types of gravitational fields, determined by the characteristics (12). For the canonical form of the matrix $\|R_{ab}\|$ we obtain expressions analogous to the expressions for the canonical form of the matrix $\|C_{ab}\|$ in the orthonormal frame (see (13), (17), and (20)), where the matrices x and y will appear instead of the matrices $\tilde{x}$ and $\tilde{y}$. Since in the orthonormal frame the conditions (15) are satisfied, the imaginary parts of the stationary curvatures $\tilde{\lambda}_s$ and λ_s coincide, i.e., the presence of matter leads to an additional contribution only to the real parts of the stationary curvatures. For spaces of types $\tilde{T}_1$ and $\tilde{T}_2$ we have

$$\tilde{\lambda}_s = \lambda_s - \frac{\varkappa}{2c^2} U_{ss} + \frac{\varkappa\rho}{6} + \frac{\varkappa U}{3c^2}. \tag{28}$$

From the canonical forms of the matrix $\|R_{ab}\|$ for all three types it follows that Einstein spaces for which, in some reference frame, the condition $Y_{ijk} = 0$ is satisfied can belong only to type T_1 with real stationary curvatures.

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Note: Figure translations are in progress. See original paper for figures.