PHYSICS

A. Z. PATASHINSKII, A. A. KHAR' KOV

1970

SovietRxiv

Abstract

Full Text

PHYSICS

A. Z. PATASHINSKII, A. A. KHAR’ KOV

ON THE GRAVITATIONAL COLLAPSE OF A WEAKLY NONSPHERICAL MASS

(Presented by Academician Ya. B. Zel̆ dovich, 23 VI 1969)

The gravitational collapse of a body whose shape differs little from spherical was studied in work (^{1}). In this case the general properties of the metric outside the body turn out to be, in the main, the same as for the collapse of a sphere, and are determined by the principal spherically symmetric static part of the metric. There is also a small nonspherical addition to the metric, decaying with time. In passing through the gravitational radius r_g in the comoving frame of reference, no field arises in it (^{2}). The aim of the present note is a more detailed consideration of the problem. It is assumed that all characteristics of the body have axial symmetry. We choose a system of units in which $c = 1$, $k = 1$, $r_g = 1$. Since the deviation from spherical symmetry is assumed small, the equation for the additions to the metric in vacuum may be taken in the linear approximation. This makes it possible at once to separate out the angular dependence of the additions, since the metric of the zeroth approximation admits the group of three-dimensional rotations O_3. The following mathematical device is convenient. We shall first consider a perturbation of the Schwarzschild metric, and then transform the solution obtained by means of the Lemaître transformation (^{3}) for the zeroth approximation. The frame of reference obtained in this way is not comoving for a test particle, but differs little from it. The symmetry of the problem permits one to seek the Schwarzschild metric in diagonal form:

$$ds^2 = \left(1 - \frac{1}{r}\right)(1+d)\,dt^2 - \frac{1+a}{1-1/r}\,dr^2 - r^2(1+b)\,d\theta^2 - r^2\sin^2\theta\,(1+c)\,d\varphi^2. \quad (1)$$

The field equations in vacuum $R_{ik} = 0$ give equations for the quantities a, b, c, d. To separate the angular dependence, we expand these functions in the system of generalized spherical functions, which for our case of independence of the angle φ have the form $u^l_{0n}(\theta) = P^l_{0n}(\cos\theta)$ (^{4}). It turns out that

$$d = \sum_{l=2} d_l P^l_{00}(\cos\theta), \qquad a = \sum_{l=2} a_l P^l_{00}(\cos\theta),$$

$$b + c = \chi = \sum_{l=2} \chi_l P^l_{00}(\cos\theta), \qquad c - b = \eta = \sum_{l=2} \eta_l P^l_{02}(\cos\theta). \tag{2}$$

For analysis of the field equations, let us consider these equations at $r - 1 = y \ll 1$. Their solutions in this region have the form; for example, for the function χ_l:

$$\chi_l = yF_+(t + \ln|y|) + yF_-(t - \ln|y|) + \varphi(t)\sqrt{y} + f(y), \tag{3}$$

where we have omitted the index l on the function; in what follows we shall consider some one value of l. In the solution (3), F_- and F_+ are respectively the retarded and advanced solutions of the field equations, $\varphi(t)$ is an arbitrary –

wave function. The function $f(y)$ has the form

$$f(y) \simeq \text{const} \cdot y \ln|y| \tag{4}$$

and is the asymptotic form of the known static solution (1). As $y \to 0$, all quantities must have no singularity; therefore $\varphi(t) = 0$. A function $f(y)$ of the form (4) can be obtained from F_+ and F_-. Thus we obtain

$$\chi_l = y\left[F_+(t + \ln|y|) + F_-(t - \ln y)\right], \tag{5}$$

$$a_l = d_l = -\frac{1}{2}\sqrt{\frac{(l-1)(l+2)}{l(l+1)}}\,\eta_l =$$

$$= \frac{1}{2}\left[F'_+ - F'_-\right] + \frac{1}{4}\left[F_+ + F_-\right].$$

We now perform the Lemaître transformation. The metric takes the form:

$$ds^2 = \left(1 + \frac{rd - a}{r - 1}\right) d\tau^2 + 2\frac{a - d}{r - 1}\, dR\, d\tau -$$

$$- \left(\frac{1}{r} - \frac{d - ra}{r(r-1)}\right) dR^2 - r^2(1 + b)d\theta^2 -$$

$$-r^2 \sin^2\theta(1 + c)d\varphi^2. \tag{6}$$

Fig. 1

All functions must then be expressed in terms of τ and R. Note that the advanced and retarded solutions written above are not wave solutions over the

Fig. 1

Figure 1: Fig. 1

entire range of values of y. In a certain sense the wave zone of the problem has turned out to be the neighborhood of the gravitational radius (in ordinary radiation problems this zone lies at large distances from the system). The region of large values of $|y|$ in our case corresponds to the quasistatic zone. Our task is to construct, in all space, a solution decreasing as $R \to \infty$. In the static case such a solution is the Erez-Rosen type solution, given in (1) and having the asymptotic form

$$\chi \sim y \ln |y|, \qquad \eta \sim \ln |y|. \tag{7}$$

For analysis of the solution we shall use Fig. 1, which shows world lines in the Lemaître picture. The world line of the boundary of the matter before entering the region $y \ll 1$ is OA. The solution in the region to the right of OA, the world line of a light ray emitted from point A, corresponds essentially to the nonrelativistic problem and is clear from general considerations.

We assume, following (1), that collapse after passing over the gravitational radius near it proceeds essentially in the same way as in the spherical case.

If it were to turn out that for $y < y_0$ up to $y = 0$ the solutions are almost static, then throughout the whole region they would coincide with Erez-Rosen type solutions, i.e., a singularity would appear at $y = 0$. The linear approximation would become incorrect, but the Erez-Rosen solution is exact. As will be seen below, this would mean the appearance of a singularity in the motion of the matter at $y = 0$, for which we see no grounds. For $|y| \ll y_0$ the solutions of our equations have the form of a superposition of retarded and advanced functions of the form

$$n(\tau, y) = f_1(\tau - 2 \ln |y|) + f_2(\tau). \tag{8}$$

Consider the metric at a point S, moving along the line $y = \text{const}$. At the moment when this point lies on the matter line OBD, the metric at it coincides with the metric on the matter, so that $n(\tau, y(\tau)) = q(\tau)$. The line of moti-

of matter near $y = 0$ is $y = \tau\sqrt{2}/2$, if $\tau = 0$ at $y = 0$. The metric $q(\tau)$ is assumed to be regular:

$$q(\tau) = n_0 + \tau n_1. \tag{9}$$

Therefore

$$f_1(-2\ln|y|) + f_2(\tau) = n_0 + \tau n_1. \tag{10}$$

We obtain

$$f_1(-2\ln|\tau|) = n_0 - f_2(0) + [n_1 - f_2'(0)]e^{-1/2(2\ln|\tau|)},$$

i.e.

$$f_1(z) = n_0 - f_2(0) + [n_1 - f_2'(0)]e^{-z/2}. \tag{11}$$

Thus,

$$n(\tau, y) = n_0 + \tau f_2'(0) + [n_1 - f_2'(0)]e^{-1/2(\tau - 2\ln|y|)}. \tag{12}$$

The metric changes slowly as one moves away from the matter. Let us now consider the case when τ is large, while $y \ll y_0$, and the point S has moved to the position K. Let us compare the metric at two points L and K, where the point L lies on the same line of advanced potential as K, but with $y < 0$, $|y| \ll y_0$. Then

$$\Delta n_{K,L} = n(\tau_K, y_K) - n(\tau_L, y_L) = f_1(\tau_K - 2\ln|y_K|) - f_1(\tau_L - 2\ln|y_L|). \tag{13}$$

The value of the retarded function is found by moving backward in time along the geodesic line going from the matter to the points K and L (see (11)):

$$\Delta n_{KL} = (n_1 - f_2'(0))(y_K - y_L)e^{-\tau_K/2}. \tag{14}$$

We thus find that for very large τ, in the region $|y| \ll 1$, the metric is constant up to the small quantity (14). Everything is determined by the behavior of the metric at $y \sim 1$ on the boundary of our wave zone.

The correct linear combination for the advanced and retarded functions must be chosen from the considerations that at spatial infinity the field should contain only an outgoing wave of gravitational radiation. Let us represent, in Schwarzschild coordinates, the corrections to the metric in the form of harmonic components in time,

$$n_\omega(t, y) = C_+(\omega)e^{i\omega(t+\ln|y|)} + C_-(\omega)e^{i\omega(t-\ln|y|)}. \tag{15}$$

First consider small frequencies $\omega \ll 1$. For these frequencies, at large distances there is a quasi-static zone of gravitational radiation. Therefore the solution for this frequency range, for small but finite y, must have the form corresponding to the Erez–Rosen solution:

$$n_\omega \simeq C(\omega) e^{i\omega t} \ln |y|. \tag{16}$$

From this we find that in formula (15), for $\omega \ll 1$,

$$C_+(\omega) = -C_-(\omega), \qquad C(\omega) = 2i\omega C_+(\omega). \tag{17}$$

In the region of very high frequencies $\omega \gg l$, where l is the number of the spherical harmonic, the wave zone at $y \ll 1$ immediately passes into the wave zone of radiation extending to spatial infinity. In this case the solution is mainly of the form of an outgoing wave, and the amplitude $C_+(\omega)$ of the advanced function in (15) is small compared with $C_-(\omega)$ by the parameter $1/\omega$.

The behavior of the retarded function was defined above by formula (11). Thus the effects of nonsphericity disappear in the external space exponentially in time (regardless of whether it is Schwarzschild or Lemaître).

The continuation of the solution into the region of large values of $|y|$ for $y < 0$, after determining the solution in a neighborhood of $y = 0$, must determine the motion of the boundary of the matter and is not considered in the present paper. The behavior of the metric in the region of small $|y|$ and for $r > r_g$ is not affected by the character of the motion of the matter after it has passed far below the gravitational radius.

Novosibirsk State
University

Received
18 VI 1969

CITED LITERATURE

1. A. G. Doroshkevich, Ya. B. Zel' dovich, I. D. Novikov, *ZhETF*, **49**, 170 (1965).
2. E. M. Lifshitz, I. M. Khalatnikov, *UFN*, **80**, 391 (1963).
3. L. D. Landau, E. M. Lifshitz, *Field Theory*, 1962.
4. I. M. Gelfand, R. A. Minlos, Z. Ya. Shapiro, *Representations of the Rotation Group and the Lorentz Group*, Moscow, 1958.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.