ON THE COVERING OF A-SETS AND MULTIPLE SEPARABILITY

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Abstract

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Problems on the covering of A-sets have been considered in a number of works by N. N. Luzin, V. I. Glivenko, P. S. Novikov, Z. I. Kozlova, and others ([1−6]). The point is that a plane A-set all of whose sections parallel to the axis OY possess a certain special property can be covered by such a B-set all of whose sections parallel to the axis OY possess analogous properties. In solving these problems, one or another separability theorem is used.

The aim of the present note is to show that the first general theorem on multiple separability ([7]) leads to a certain theorem on the covering of A-sets, encompassing a number of results that were earlier established independently of one another. The same method, but only with the use of the second general theorem on multiple separability, leads to the establishment of certain new properties of plane A-sets.

We shall consider subsets of the Baire space I_{xy}. Denote by $\delta_{n_1,\dots,n_k}$ the Baire intervals of the space I_y. Let H be some hereditary class of linear sets and $E \subset I_{xy}$. Let $\operatorname{Pr}_x E$ denote the projection of the set E onto the space I_x, and let E^x be the complete preimage of the point x under this projection. Put

$$E_y^H = \bigcup_{\substack{x \\ E^x \in H}} E^x.$$

By $\bar{U}$ we denote the closure of the set U. Put

$$\widetilde{E}_y = \bigcup_x \overline{E^x}.$$

Basic construction. Let $E \subset I_{xy}$; put

$$E_{n_1\cdots n_k} = E \cap (I_x \times \delta_{n_1\cdots n_k}), \qquad U_{n_1\cdots n_k} = \operatorname{Pr}_x E_{n_1\cdots n_k},$$
$$V_{n_1\cdots n_k} = E_{n_1\cdots n_k} \times \delta_{n_1\cdots n_k}.$$

We note that

$$\widetilde{E}_y = \bigcap_k \bigcup_{n_1\dots n_k} V_{n_1\cdots n_k}, \qquad \operatorname{Pr}_x E = A\{U_{n_1\cdots n_k}\}.$$

Let

$$A^H\{B_{n_1\cdots n_k}\}$$

denote the δs-operation on sets numbered by all possible tuples of natural numbers, whose base consists of all chains, each of which is the union of a closed set of A-chains that is not the closure of a set possessing the property H.

Suppose that there exists a system of sets $\{S_{n_1\ldots n_k}\}$ such that

$$S_{n_1\cdots n_k} \supset U_{n_1\cdots n_k} \Big/ A^H_{m_1\ldots m_s}\{U_{n_1\cdots n_k\, m_1\cdots m_s}\},$$

$$A^H\{S_{n_1\cdots n_k}\} = \emptyset.$$

Then, if

$$G_{n_1\cdots n_k} = S_{n_1\cdots n_k} \times \delta_{n_1\cdots n_k},$$

$$G = \bigcap_k \bigcup_{n_1\ldots n_k} G_{n_1\cdots n_k},$$

then all sets of the form G^x are closures of sets of the class H, and if $E^{x_0} \in H$, then $\overline{E}^{x_0} \subset G$.

The construction described leads to the following theorems:

The second theorem on the covering of A-sets. Let H be a hereditary class of sets such that the δs-operation $A^H\{B_{n_1\cdots n_s}\}$ and all its truncated operations

$$\overset{H}{_{m_1\cdots m_s}}A\{B_{n_1\cdots n_k\, m_1\cdots m_s}\}$$

are weaker than the A-operation. (This means that they are not stronger than the A-operation and are comparable with it.) Then, whatever the A-set $E \subset I_{xy}$ may be, there always exists a CA-set G, all whose subsets of the form G^x are closures of sets of class H, and which contains all sets of the form E^x belonging to the class H.

For the proof of this theorem it suffices to refer to the fact that, by virtue of the second theorem on multiple separability, under the conditions of the present theorem one may assume that all the sets $S_{n_1\cdots n_k}$ are CA-sets. Hence it follows at once that G is also a CA-set.

Similarly one proves

The first theorem on the covering of A-sets. Suppose that, under all the conditions of the preceding theorem, all sets of the form $E^x \in H$.

Then there exists a B-set $G \supset E$ such that all sets of the form G^x are closures of sets of class H.

For the proof it is enough to observe that, under the conditions of this theorem,

$$A^H\{U_{n_1\cdots n_k}\} = \emptyset,$$

and therefore, according to the first theorem on multiple separability, there exist B-sets $S_{n_1\cdots n_k} \supset U_{n_1\cdots n_k}$ such that

$$A^H\{S_{n_1\cdots n_k}\} = \emptyset.$$

In this case the B-set $G \supset E$ and all sets of the form G^x are closures of sets of class H.

Both theorems discussed here are applicable, in particular, to the following properties H.

1. To contain not more than n points (n natural).
2. To be a completely ordered set (in this case the sets G^x are closed completely ordered sets).
3. To be a completely ordered set of type $< \alpha$; in this case the sets G^x are closed completely ordered sets of type $< \alpha$, if α is a number of the first kind, and of type $\leqslant \alpha$, if α is a number of the second kind.
4. To be a countable reducible set. In this case the sets G^x are closed countable reducible sets.
5. To be a reducible set of rank $< \alpha$; in this case the sets G^x are countable closed reducible sets of rank $< \alpha$, if α is a number of the first kind, and of rank $\leqslant \alpha$, if α is a number of the second kind.
6. To have compact closure; in this case the set G^x is compact.

We note that in the cases where the class H is the class of countable sets, of sets of type F_σ, and also of sets absolutely of class I, the theorems proved give nothing.

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Note: Figure translations are in progress. See original paper for figures.