ON THE ANALYTIC STRUCTURE OF SUBSPACES GENERATED BY RANDOM HOMOGENEOUS FIELDS

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Abstract

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MATHEMATICS

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ON THE ANALYTIC STRUCTURE OF SUBSPACES GENERATED BY RANDOM HOMOGENEOUS FIELDS

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1. Consider a homogeneous random field $\xi(\mathbf{t})$, $\mathbf{t} \in R^n$, with spectral density $f(\mathbf{x})$, $\mathbf{x} \in R^n$. Let $M \subset R^n$ be some closed set of values of the time parameter $\mathbf{t}$. With the field under consideration one naturally associates the space $H(M)$—the closed linear span, in the mean-square sense, of the random variables $\xi(\mathbf{t})$ such that $\mathbf{t} \in M$. The linear space $H(R^n)$ is a Hilbert space with inner product $\langle \xi, \eta \rangle = \mathbf{E}\xi\eta$. The spaces $H(M)$, $M \subset R^n$, are its subspaces. In the case $n = 1$, the question of the structure of the spaces $H(M)$, when M is an interval, was completely solved by M. G. Krein. His results are formulated in the paper (^{1}). Related questions are also investigated in the paper of Levinson and McKean (^{2}) (the work of M. G. Krein (^{1}), apparently, remained unknown to the authors of (^{2})).

We shall study the structure of subspaces in the case where M is a rectangle:

$$M = K^\rho = \{\mathbf{t} = (t_1, \dots, t_n) : \ -\rho_j \le t_j \le \rho_j, \ j = \overline{1, n}; \ \rho = (\rho_1, \dots, \rho_n)\}.$$

As is known (^{3}), there exists an isometric correspondence between the space $H(R^n)$ and the space $L^2(R^n, f)$ of functions square-summable with weight $f(\mathbf{x})$. Under this correspondence, the subspace $H(M)$ corresponds to the subspace $L^2(M, f)$, which is the closure in $L^2(R^n, f)$ of the linear span of the set of functions $\exp i(\mathbf{x}, \mathbf{t})$ such that $\mathbf{t} \in M$. Here $(\mathbf{x}, \mathbf{t}) = x_1 t_1 + \dots + x_{nt} n$. Thus, the study of the structure of the subspace $H(K_\rho)$ is reduced to the study of $L^2(K_\rho, f)$.

2. Introduce the notation:

$$f^*_{t_0}(t) = \inf_{t_0^2 \le |\mathbf{x}|^2 \le t^2} f(\mathbf{x}), \qquad |\mathbf{x}|^2 = x_1^2 + \dots + x_n^2,$$

$$g^*(t) = \inf_{|\mathbf{x}|^2 = t^2} f(\mathbf{x}),$$

$S(M, \varepsilon)$ is the closure of the ε-neighborhood of the set M.

$$L^2(M^+, f) = \bigcap_{n=1}^{\infty} L^2(S(M, \varepsilon_n), f), \qquad \text{where } \varepsilon_n \downarrow 0.$$

Theorem 1. *If for some $t_0 \geq 0$ the inequality*

$$\int_{t_0}^{\infty} \frac{\ln f^*_{t_0}(t)}{1+t^2}\, dt > -\infty, \tag{1}$$

holds, then the set of functions $L^2(K^+_\rho, f)$ coincides with the set of entire analytic functions $\varphi(z)$, $z \in C^n$, satisfying the conditions:

$$\int_{R^n} |\varphi(\mathbf{x})|^2 f(\mathbf{x})\, d\mathbf{x} < \infty \tag{2}$$

and

$$\varphi(z) \text{ is an entire function of type } \rho = (\rho_1, \dots, \rho_n) \tag{3}$$

(the definition of the type of an entire function here is the same as in ([4])).

We denote by L°_ρ the set of entire functions satisfying (2) and (3).

In the proof of the theorem Bernstein' s function is used, which is defined (see ([5]), p. 376) as follows. Let a sequence of numbers $t_k > 0$, $k = 1, 2, \dots$, be given such that

$$p = \pi \sum_{k=1}^{\infty} \frac{1}{t_k} < \infty.$$

Let $a_k^2 = \dfrac{1}{p^2 t_k^2}$. From the sequence $\{a_k\}$ we construct the Bernstein function

$$B(z) = \prod_1^{\infty} \sin^2 \frac{\pi}{2} \sqrt{1 + a_k^2 z^2/(1 + a_k^2 z^2)}.$$

From the same sequence $\{a_k\}$ one can construct a function of n variables

$$B(z) = B(z_1, \dots, z_n) =$$

$$= \prod_{1}^{\infty} \sin^2 \frac{\pi}{2} \sqrt{1 + a_k^2(z_1^2 + ... + z_n^2/[1 + a_k^2(z_1^2 + ... + z_n^2)])}.$$

$B(z)$ and $B(z_1, ..., z_n)$ are entire analytic functions of finite degree. The type of each of them is 1. The proof of the theorem is based on the following lemmas.

Lemma 1. Let $u(t) > 0,\ t \in (0, \infty)$, be a monotonically increasing function satisfying the conditions:

$$\int_0^\infty \frac{\ln u(t)}{1+t^2}\, dt < +\infty \qquad u(t) \xrightarrow[t\to+\infty]{} +\infty.$$

Then there exists a sequence $t_k > 0,\ k = 1, 2, ...$, such that

$$\sum_{k=1}^{\infty} \frac{1}{t_k} < \infty,$$

and the Bernstein function $B(z)$ constructed from it ($z = t+is$) has the property: for every $\lambda \neq 0$,

$$u(t)B(\lambda t) \leq c(\lambda),$$

where $c(\lambda)$ is a constant depending on λ.

This lemma is a consequence of a theorem proved by O. I. Inozemtsev and V. A. Marchenko in (^{6}). Put $f^*_{t_0=0}(t) = f^*(t)$. For $t_0 = 0$ the function $1/\sqrt{f^*(t)}$ satisfies the conditions of Lemma 1; therefore the following lemma makes sense.

Lemma 2. Let $B(z)$ be such a Bernstein function that for every $\lambda \neq 0$

$$B(\lambda t)/\sqrt{f^*(t)} \leq c(\lambda),$$

where $c(\lambda)$ is a constant depending on λ. Then one can construct a Bernstein function $B(z_1, ..., z_n)$ such that, whatever $\lambda_1, ..., \lambda_n \neq 0$ may be,

$$B^2(\lambda_1 x_1, ..., \lambda_{nx} n)/f(x_1, ..., x_n) \leq d(\lambda_1, ..., \lambda_n),$$

where $d(\lambda_1, ..., \lambda_n)$ is a constant depending on $\lambda_1, ..., \lambda_n$.

Lemma 3. Let $\varphi(z_1, ..., z_n)$ be an entire function of finite degree, and let

$$\varphi(z_1, ..., z_n)B(\lambda_1 z_1, ..., \lambda_{nz} n)$$

have type

$$\sigma = (\rho_1 + \max(\lambda_j,\ j = 1, n), \dots, \rho_n + \max(\lambda_j,\ j = 1, n)).$$

Then the type of the function $\varphi(z_1, \dots, z_n)$ is equal to $\rho = (\rho_1, \dots, \rho_n)$.

3. Theorem 2. Let the function $g^*(t)$ satisfy the conditions:

a) $g^*(t)$ is continuous,

b) $g^*(t_1 + t_2) \geq c\, g^*(t_1) g^*(t_2)$;

c)

$$\int_0^\infty \frac{\ln g^*(t)}{1 + t^2}\, dt > -\infty; \tag{4}$$

then the set of functions $L^2(K_\rho^+, f)$ coincides with L_ρ°.

The proof of the theorem is analogous to the proof of Theorem 1.

4. In the paper of V. N. Tutubalin and M. I. Freidlin [7] it is proved for a random stationary process that if its spectral density $f(x) \geqslant 1/|x|^p$ for some $p > 0$ and all x such that $|x| > x_0 > 0$ (x_0 is any positive number), then the functions $1, ix, \dots, (ix)^k$ form a basis in the space $L^2(K_0^+, f)$ (k is the number of derivatives possessed by the trajectories of the process). The following is connected with their result.

Theorem 3. Let the spectral density $f(x)$ satisfy condition (1) or conditions (4). Let $\{M_n\}_{n=1}^\infty$ be a sequence of bounded closed sets for which the point $\{O\} = \{t_1 = 0, \dots, \dots, t_n = 0\}$ is an interior point, and

$$\bigcap_{n=1}^\infty M_n = \{O\}.$$

Then

$$\bigcap_{n=1}^\infty L^2(M_n^+, f) = L^0.$$

The proof of the theorem uses Lemma 2; its central part is the proof of the following lemma:

Lemma 4. Let M_1 and M_2 be two bounded closed sets for which the point O is an interior point, and let $f(x)$ satisfy condition (1) for $t_0 = O$ or conditions (4); then

$$L^2(M_1^+, f) \cap L^2(M_2^+, f) = L^2(M_1^+ \cap M_2^+, f).$$

Corollary. If the conditions of the last lemma are satisfied, then there exists no closed bounded set M such that

$$L^2(M^+, f) = L^2(R^n, f).$$

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Note: Figure translations are in progress. See original paper for figures.

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