SYMMETRIC DOMAINS AND JORDAN ALGEBRAS

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 519.46

MATHEMATICS

V. I. SEMYANISTYI

SYMMETRIC DOMAINS AND JORDAN ALGEBRAS

(Presented by Academician I. G. Petrovskii on 13 VI 1969)

In the present note there is proposed what seems to the author a natural generalization, to the real case, of I. I. Pyatetskii-Shapiro' s construction of Siegel domains ([1]), and some results are announced concerning the case when the proposed construction gives the standard realization of a broad class of Riemannian symmetric spaces of noncompact type (this class contains all irreducible spaces with the exception of thirteen).

We agree that, unless the contrary is stated, all vector spaces and algebras are considered over the field of real numbers.

1. Let $\mathfrak{A}$ and $\mathfrak{X}$ be vector spaces and let $*$ be some involution in $\mathfrak{A}$. A bilinear mapping $F : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{A}$ is called **Hermitian relative to** $*$ if $F(x, y)^* = F(y, x)$ for all $x, y \in \mathfrak{X}$. Put $\mathfrak{A}^+ = \{a \in \mathfrak{A} : a = a^*\}$, $\mathfrak{A}^- = \{a \in \mathfrak{A} : a = -a^*\}$. Suppose that in $\mathfrak{A}^+$ a convex cone V is given which contains no straight lines. A Hermitian mapping F is called V-**definite** if $F(x, x) \in \overline{V}$ for all $x \in \mathfrak{X}$ (the bar denotes the operation of closure) and if from $F(x, x) = 0$ it follows that $x = 0$. Suppose, finally, that in $\mathfrak{A}$ there acts a certain group S of affine transformations commuting with $*$, transitive in V (so that V is homogeneous, cf. ([2])). The mapping F is called **homogeneous relative to** S if in $\mathfrak{X}$ there is given some representation of the group S and ${}^{s}F(x, y) = F(sx, sy)$ for all $x, y \in \mathfrak{X}$, $s \in S$. We note that in this case $TF(x, y) = F(Tx, y) + F(x, Ty)$ for all $T \in \mathfrak{f}$, where $\mathfrak{f}$ denotes the Lie algebra of the group S.

Definition. Let F be a Hermitian V-definite mapping $\mathfrak{X} \times \mathfrak{X}$ into $\mathfrak{A}$, and let $\mathfrak{P} = \mathfrak{A} \oplus \mathfrak{X}$. The set $\mathcal{P} = \{a \oplus x \in \mathfrak{A} \oplus \mathfrak{X} : \ a + a^* - F(x, x) \in V\}$ is called the **Siegel domain** associated with the mapping F, the involution $*$, and the cone V. If, moreover, F is homogeneous, then the Siegel domain is also called **homogeneous**.

The last term is justified by the following.

Theorem 1. *Every homogeneous Siegel domain is affinely homogeneous.*

The latter means that some group R of affine transformations acts transitively in it. We note that the Lie algebra $\mathfrak{r}$ of this group, considered as an algebra of infinitesimal affine transformations, acts on $p = a \oplus x$ in the following way:

$$a \mapsto Ta - F(u,x) + c; \qquad x \mapsto Tx + u \qquad (T \in \mathfrak{f},\ u \in \mathfrak{X},\ c \in \mathfrak{A}^-),$$

so that the space $\mathfrak{r}$ coincides with $\mathfrak{f} \oplus \mathfrak{X} \oplus \mathfrak{A}^-$.

Let us note that if $\mathfrak{A}$ and $\mathfrak{X}$ are vector spaces over the field of complex numbers and $*$ denotes passage to the complex-conjugate coordinates, then we obtain exactly the complex Siegel domains considered in ([1]). Further, if $\mathfrak{A}^+ = \mathfrak{A}$ and $*$ is the identity transformation, then, as was shown by E. B. Vinberg ([2]), Siegel domains in this case realize all affinely homogeneous domains containing no straight lines.

The definition given above does not exclude the case $\mathfrak{X} = \{0\}$, $F(x,y) = 0$. By analogy with the definitions adopted in ([1]), such domains are naturally called **Siegel domains of the first kind**, while domains for which $\mathfrak{X} = \{0\}$ are **Siegel domains of the second kind**.

2. Let us consider one specialized case of Siegel domains. Suppose that $\mathfrak{A}$ is a semisimple Jordan algebra, the multiplication law in which is denoted by a dot, $\mathfrak{A}^+$ is its maximal compact subalgebra, and $*$ is an involutive automorphism of the algebra $\mathfrak{A}$ whose set of fixed elements is $\mathfrak{A}^+$. By R denote the regular representation of the algebra $\mathfrak{A}$: $R_a b = a \cdot b$ $(a, b \in \mathfrak{A})$. Let V be the interior of the set $\{a^2,\ a \in \mathfrak{A}\}$. As is known ([3]), V is a self-adjoint cone, homogeneous with respect to the (connected) group S generated by the Lie algebra

$$\mathfrak{f} = R(\mathfrak{A}^+) \oplus [R(\mathfrak{A}^+), R(\mathfrak{A}^+)].$$

A V-definite Hermitian mapping $F : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{A}$ is called homogeneous with respect to the algebra $\mathfrak{A}$ if $\mathfrak{X}$ is an $\mathfrak{A}$-module, i.e. if a general Jordan representation L of the algebra $\mathfrak{A}$ is given in $\mathfrak{X}$ (for definitions see ([6])) and for all $x, y \in \mathfrak{X},\ a \in \mathfrak{A}$

$$R_{aF}(x,y) = F(L_a x, y) + F(x, L_a y).$$

Proposition 1. *If the Hermitian mapping F is homogeneous with respect to $\mathfrak{A}$, then it is also homogeneous with respect to S.*

This proposition makes it possible, with the aid of a V-definite Hermitian mapping F homogeneous with respect to $\mathfrak{A}$, to construct a homogeneous Siegel domain $\mathcal{P}$. We shall call it a **Siegel domain associated with the semisimple Jordan algebra $\mathfrak{A}$**.

Theorem 2. *A Siegel domain associated with a semisimple Jordan algebra $\mathfrak{A}$ is a symmetric Riemannian space of noncompact type.*

The proof of this theorem is based on the construction of a semisimple Lie algebra $\mathfrak{g}$, generating a transitive group G of transformations of the Siegel domain $\mathcal{P}$. This construction is carried out differently for Siegel domains of the first and of the second kind. For domains of the first kind it was in fact carried out in ([4]) (see also ([5]), where the same idea is developed in greater detail). Let us note that the Siegel domain does not coincide with the model constructed in ([4]), but can be transformed into it by a transformation of the form $\exp X$, where X is some element of the Lie algebra of the family D^2 generated by $\mathfrak{A}$. Here we shall outline the scheme for constructing the Lie algebra $\mathfrak{g}$ for a Siegel domain of the second kind. First of all, the following holds.

Proposition 2. *If the Hermitian mapping F is homogeneous with respect to $\mathfrak{A}$ and if $F(x, x) \neq 0$ for at least one $x \in \mathfrak{X}$, then $\mathfrak{A}$ is a special Jordan algebra and L is its special representation.*

Thus, one may assume that $a \cdot b = ab + ba$, where the product ab is associative. Put

$$T_b(a \oplus x) = ba \oplus L_{bx}.$$

Then T is a special representation of $\mathfrak{A}$ in $\mathfrak{B} = \mathfrak{A} \oplus \mathfrak{X}$. To each pair $z_1 = a_1 \oplus x_1,\ z_2 = a_2 \oplus x_2$ of elements of $\mathfrak{B}$ we associate transformations of the space $\mathfrak{B}$:

$$\varphi(z_1, z_2) = T_{a_1} T_{a_2^*} + T_{F(x_1, x_2)}, \qquad \Phi(z_1, z_2)z = \varphi(z_1, z_2)z + \varphi(z, z_2)z_1.$$

The Lie algebra generated by all $\Phi(z_1, z_2),\ z_1, z_2 \in \mathfrak{B}$, will be denoted by $\mathfrak{q}$. Introduce in $\mathfrak{q}$ the anti-involution $\#$, putting $\Phi(z_1, z_2)^{\#} = \Phi(z_2, z_1)$, and let

$$\mathfrak{u}' = \{Y \in \mathfrak{q} :\ Y = -Y^{\#}\}.$$

Then the commutation operation in $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{B}$, defined by the rule

$$[Y_1 \oplus z_1, Y_2 \oplus z_2] = Y \oplus z,$$

where

$$Y = [Y_1, Y_2] + \Phi(z_1, z_2) - \Phi(z_2, z_1), \qquad z = Y_1 z_2 - Y_2 z_1,$$

turns $\mathfrak{g}$ into the desired Lie algebra.

Let us note that the mapping $\Phi : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{q}$ (in the case of a Siegel domain of the first kind, when $\mathfrak{B} = \mathfrak{A}$, it is defined by the equality $2\Phi(a, b) = R_{a \cdot b^*} + [R_a, R_{b^*}]$) generally plays an important role in the geometry of the spac–

properties $\mathcal{P}$. With its help one can obtain convenient expressions for the Riemannian metric in $\mathcal{P}$, the curvature, and so on. Further, the operation $z_1 \cdot z_2 = \Phi(z_1, z)z_2$ (z fixed) turns $\mathfrak{B}$ into a Jordan algebra (without identity in the case of a Siegel domain of the second kind). The mapping Φ makes it possible to embed $\mathfrak{B}$ in a certain Lie algebra in the same way in which, for a Jordan algebra with identity, the Lie algebra of the family D^2 is constructed (see ([5,7])).

3. A natural question arises: what are those symmetric spaces which admit models in the form of Siegel domains? It turns out that the answer admits a simple and natural formulation in terms of the roots of the symmetric space.

Let $P = G/U$ be a Riemannian symmetric space of noncompact type, $\mathfrak{g}$ the Lie algebra of the group G, $\mathfrak{u}$ the subalgebra corresponding to U, $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ the Cartan decomposition of the algebra $\mathfrak{g}$, and σ an antiautomorphism of the algebra $\mathfrak{g}$ such that $\mathfrak{u} = \{X \in \mathfrak{g} : X = -\sigma X\}$, $\mathfrak{p} = \{X \in \mathfrak{g} : X = \sigma X\}$. A Cartan subalgebra $\mathfrak{h}$ of the space P is defined as a maximal Abelian subalgebra contained in $\mathfrak{p}$. Its dimension is called the rank of the space P. In the usual way, roots and root subspaces relative to $\mathfrak{h}$ are defined. They possess all the properties of the root systems of complex semisimple algebras, on which the well-known classification of irreducible root systems into the classes A_n, B_n, C_n, D_n (classical systems), E_6, E_7, E_8, F_4, G_2 (exceptional systems) is based. Let us note here that the class of the root system of an irreducible symmetric space does not, generally speaking, coincide with the class of the root system of the complex extension of its group of motions.

Proposition 3. If the root system of a symmetric space is classical, then in the Cartan subalgebra of this space there exists an element E such that:

1°. $\operatorname{ad} E$, as eigenvalues, can have only $0,\ \pm 1/2,\ \pm 1$.

2°. If $\mathfrak{g}^k$ is the eigenspace of $\operatorname{ad} E$ corresponding to the eigenvalue $k/2$, then the decomposition

$$\mathfrak{g} = \bigoplus_{k=-2}^{2} \mathfrak{g}^k$$

is a grading of the algebra $\mathfrak{g}$:

$$[\mathfrak{g}^k, \mathfrak{g}^l] \subset \mathfrak{g}^{k+l}$$

(it is assumed that $\mathfrak{g}^m = \{0\}$ for $|m| > 2$), and moreover $\sigma\mathfrak{g}^k \subset \mathfrak{g}^{-k}$.

3°. The subspace $V = \exp(\mathfrak{g}^0 \cap \mathfrak{p})$ is a self-dual convex cone, and this is the maximal cone contained in P.

It follows from 3° that the subspace $\mathfrak{A}^+ = \mathfrak{g}^0 \cap \mathfrak{p}$ can be endowed with the structure of a semisimple compact Jordan algebra (the multiplication in it will be denoted by $a^+ \cdot b^+$; $a^+, b^+ \in \mathfrak{A}^+$). Put $\mathfrak{A}^- = \mathfrak{g}^2$, $\mathfrak{A} = \mathfrak{A}^+ \oplus \mathfrak{A}^-$, and define in $\mathfrak{A}$ the operation

$$(a_1^+ \oplus a_1^-) \cdot (a_2^+ \oplus a_2^-) = a^+ \oplus a^-,$$

$$a^+ = a_1^+ \cdot a_2^+ + [a_1^-, {}^\sigma a_2^-] + [a_2^-, {}^\sigma a_1^-], \qquad a^- = [a_1^+, a_2^-] + [a_2^+, a_1^-].$$

Proposition 4. The subspace $\mathfrak{A} \subset \mathfrak{g}$ with the operation just defined is a semisimple Jordan algebra, in which $\mathfrak{A}^+$ is a maximal compact subalgebra.

Put now $\mathfrak{X} = \mathfrak{g}^1$,

$$F(x, y) = \frac{1}{2}\{[x, \sigma y] + [y, \sigma x]\} + [x, y] \qquad (x, y \in \mathfrak{X}).$$

Proposition 5. The mapping $F : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{A}$ is Hermitian, V-definite, and homogeneous with respect to $\mathfrak{A}$. The Siegel domain $\mathcal{P}$ associated with $\mathfrak{A}$ is isomorphic to the original symmetric space P.

The final result is formulated as follows.

Theorem 2. Every Riemannian symmetric space of noncompact type with a classical root system admits a model in the form of

Siegel domain and thereby is a symmetric affinely homogeneous domain. Here the following Siegel domains correspond to irreducible rank spaces greater than one:

to the class A_n of the root system—Siegel domains of the second kind associated with compact simple Jordan algebras (these are affinely homogeneous domains containing no lines);

to the class B_n—Siegel domains of the second kind associated with noncompact simple Jordan algebras;

to the class C_n—complex or quaternionic Siegel domains of the first kind associated with simple Jordan algebras;

to the class D_n—Siegel domains of the first kind associated with simple Jordan algebras that do not admit a complex or quaternionic structure.

Let us note that in the class B_n the criterion for the existence of a complex or quaternionic structure is the simultaneous presence in the root system of roots α and 2α.

It is very likely that symmetric spaces with exceptional root systems are not domains; however, the author has not been able to prove this fact.

The author thanks P. K. Rashevskii and the participants of his seminar for discussing the present work.

Kolomna Pedagogical
Institute

Received
11 V 1969

REFERENCES

1. I. I. Pyatetskii-Shapiro, *Geometry of Classical Domains and the Theory of Automorphic Functions*, Moscow, 1961.

2. B. Vinberg, Tr. Moskov. Mat. Obshch., **12**, 304 (1963).

3. E. B. Vinberg, DAN, **133**, No. 1, 9 (1960).

4. I. L. Kantor, A. I. Sirota, A. S. Solodovnikov, DAN, **173**, No. 3, 511 (1967).

5. M. Koecher, Am. J. Math., **89**, 3, 787 (1967); **90**, 2, 476 (1968).

6. P. Cohn, *Universal Algebra*, Moscow, 1968.

7. I. L. Kantor, Tr. Seminara po vektorn. i tenzorn. analizu, **13**, 310 (1966).

Note: Figure translations are in progress. See original paper for figures.