ON THE STABILITY OF RELAXATION PROCESSES

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Abstract

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ON THE STABILITY OF RELAXATION PROCESSES

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Let $\varphi(x)$ be a functional in a real Hilbert space, defined in some convex domain W. Suppose that $\varphi(x)$ has in W a derivative $F(x)$ and

$$m\|y\|^2 \leq (F(x+y) - Fx, y) \leq M\|y\|^2, \tag{1}$$

where $M \geq m > 0$ are constants. We denote the minimum point of the functional by x^0 and put $\Delta(x) = \varphi(x) - \varphi(x^0)$. An operator $\Gamma : W \to W$ will be called relaxation* if $\varphi(\Gamma x) \leq \varphi(x)$ for all $x \in W$.

Consider a process of the form

$$x_{k+1} = \Gamma_k x_k \qquad (k = 0, 1, 2, ...), \tag{2}$$

where $\{\Gamma_k\}$ is a sequence of relaxation operators. We shall be interested in the behavior of the perturbed process

$$z_{k+1} = \Gamma_k z_k + v_k \qquad (k = 0, 1, 2, ...), \tag{3}$$

where $\{v_k\}$ is noise. By convergence of a process we shall mean its convergence to the point x^0, which, by virtue of (1), is equivalent to

$$\lim_{k\to\infty} \Delta(x_k) = 0.$$

We shall say that *linear convergence* takes place if

$$\overline{\lim}_{k\to\infty} \frac{\Delta(z_{k+1})}{\Delta(z_k)} < 1.$$

Everywhere below the family of operators $\{\Gamma_k\}$ is assumed to be *uniformly relaxation* in the sense that

$$S(\varepsilon) \equiv \sup_k \sup_{\Delta(x)\geq\varepsilon} \frac{\Delta(\Gamma_k x)}{\Delta(x)} < 1 \qquad (\varepsilon > 0) \tag{4}$$

(this condition is introduced by us only to simplify the formulations). The process (2) is relaxation, i.e. $\varphi(x_{k+1}) \leq \varphi(x_k)$ $(k = 0, 1, 2, ...)$. The process (3) may already fail to be relaxation.

Consider the ray $\Gamma_k z_k + \lambda v_k$ $(\lambda \geq 0)$ and denote by $\tilde{z}_k$ the point of its intersection with the level surface $\varphi(x) = \varphi(x_k)$. Put $\tilde{v}_k = \tilde{z}_k - \Gamma_k z_k$. We shall characterize the magnitude of the perturbation v_k by the ratio

$$\rho_k = \|v_k\| : \|\tilde{v}_k\|.$$

The relaxation property of the process (3) is, obviously, equivalent to the inequality $\rho_k \leq 1$ $(k = 0, 1, 2, ...)$.

Theorem 1. *Let the perturbed process be relaxation. Then, for its convergence, it is necessary and sufficient that*

$$\sum_{k=0}^{\infty}(1 - \rho_k) = \infty. \tag{5}$$

* With respect to the functional φ.

To prove Theorem 1, note that Taylor' s formula implies

$$\Delta(z_{k+1}) - \Delta(z_k) = -(1 - \rho_k)\{\Delta(z_k) - \Delta(\Gamma_k z_k) + a_k \rho_k \|\tilde{v}_k\|^2\}, \tag{6}$$

where $\frac{1}{2}m \leq a_k \leq \frac{1}{2}M$. If the process does not converge, then by (4) there exists $\mu < 1$ such that $\Delta(\Gamma_k z_k) \leq \mu\Delta(z_k)$ $(k = 0, 1, 2, ...)$. Therefore

$$\Delta(z_{k+1}) - \Delta(z_k) \leq -(1 - \rho_k)(1 - \mu)\Delta(z_k) \qquad (k = 0, 1, 2, ...), \tag{7}$$

whence, if (5) is satisfied, convergence follows. The contradiction proves the sufficiency of condition (5). On the other hand,

$$\|\tilde{v}_k\|^2 \leq 8m^{-1}\Delta(z_k), \tag{8}$$

whence

$$\Delta(z_{k+1}) - \Delta(z_k) \geq -(1 - \rho_k)(1 + 4h)\Delta(z_k), \tag{9}$$

where $h = Mm^{-1}$ is the condition number. From (7) follows the necessity of condition (5).

Theorem 2. Let there exist a constant $\mu < 1$ such that

$$\Delta(\Gamma_k x) \le \mu \Delta(x) \qquad (k = 0, 1, 2, ...). \tag{10}$$

Then, for the linear convergence of the perturbed process* it is necessary and sufficient that

$$\overline{\lim_{k\to\infty}} \rho_k < 1. \tag{11}$$

This result follows immediately from estimates (7), (9). We note that condition (11), as a sufficient condition, can be replaced by the more effective condition

$$\|v_k\| \le \frac{1-\sqrt{\mu}}{M} \|Fz_k\| \qquad (k = 0, 1, 2, ...).$$

Until now we have measured the noise level in terms of relative error. Let us now consider the question of the influence of noise that is small in the sense of absolute error. In this case the relaxational nature of the perturbed process can no longer be assumed.

We shall say that the process $\{z_k\}$ **converges with accuracy up to** ε if

$$\overline{\lim_{k\to\infty}} \Delta(z_k) \le \varepsilon.$$

Put

$$\delta_0(\varepsilon) = \sqrt{\varepsilon/2M}\left(1 - \sqrt{S(\varepsilon/4)}\right), \qquad \delta_1(\varepsilon) = 2\sqrt{2\varepsilon/m}.$$

Theorem 3. In order that the perturbed process converge with accuracy up to ε, it is necessary that

$$\overline{\lim_{k\to\infty}} \|v_k\| \le \delta_1(\varepsilon),$$

and sufficient that

$$\overline{\lim_{k\to\infty}} \|v_k\| < \delta_0(\varepsilon).$$

The necessity follows from estimate (8). To prove sufficiency, take such a $\rho < 1$ that

$$\|v_k\| \le \rho \delta_0(\varepsilon), \tag{12}$$

starting from $k = k_0$. Then ρ_k can be estimated. Namely, if $\Delta(z_k) > \frac{1}{4}\varepsilon$, then $\rho_k \le \rho$. Therefore it follows from Theorem 1 that $\Delta(z_k) \le \varepsilon/4$ for some $n \ge k_0$. By virtue of (12) and the inequality

$$\Delta(z_{k+1}) \le \Delta(z_k) + \sqrt{2M\Delta(z_k)}\|v_k\| + \frac{1}{2}M\|v_k\|^2$$

* Still under the assumption that it is relaxational.

there is the implication

$$\Delta(z_k) \le \varepsilon/4 \Rightarrow \Delta(z_{k+1}) \le \varepsilon.$$

Consequently, $\Delta(z_k) \le \varepsilon$ $(k \ge n)$.

Remark. Under condition (10) one may set

$$\delta_0(\varepsilon) = (1 - \sqrt{\mu})\sqrt{2\varepsilon/M}, \qquad \delta_1(\varepsilon) = (1 + \sqrt{\mu})\sqrt{2\varepsilon/m}.$$

Thus, the resulting error of the process has exactly the same order as the error of each step.

Corollary 1. *If condition (10) is satisfied, then there exist constants c_0, c_1, $(0 < c_1 < c_0)$, such that*

$$c_1 \varlimsup_{k\to\infty} \|v_k\| \le \varlimsup_{k\to\infty} \|x_k - x^0\| \le c_0 \varlimsup_{k\to\infty} \|v_k\|.$$

One may set

$$c_1 = 1/(\sqrt{\mu} + 1)\sqrt{h}, \qquad c_0 = \sqrt{h}/(1 - \sqrt{\mu}).$$

Corollary 2. *For convergence of the perturbed process it is necessary and sufficient that*

$$\lim_{k\to\infty} v_k = 0,$$

The general theorems obtained can be applied to the investigation of the stability of concrete processes. The stability of various concrete processes was studied in works ([1]–[10]).

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Note: Figure translations are in progress. See original paper for figures.