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Abstract

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MATHEMATICS

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A POSTERIORI AND WEIGHTED ERROR ESTIMATES FOR SOLUTIONS OF THE POISSON EQUATION AND THEIR DERIVATIVES COMPUTED BY THE GRID METHOD

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A posteriori error estimates are given for solutions by the grid method of the Dirichlet problem for the Laplace and Poisson equations, expressed in terms of the approximate solution and containing only known quantities. These estimates have the unimprovable second order with respect to the mesh size h ([1]) and are obtained under smoothness requirements on the right-hand side of the equation, the boundary, and the boundary values that cannot be substantially reduced ([2], [3]). Error estimates are obtained for solutions and their derivatives of any order, computed from the approximate solution, depending on the weight $\rho + h$, where ρ is the distance from the boundary of the domain.

1. Consider the boundary-value problem

$$\Delta u = f \text{ on } \Omega, \qquad u = \varphi \text{ on } \gamma, \tag{1}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$; γ is the boundary of a finite two-dimensional domain Ω; f and φ are given functions. We shall say that $\gamma \in C_{k,\lambda}$, $k \geq 2$, if the $(k-2)$-nd derivative of the curvature of γ is continuous and satisfies a Hölder condition of exponent λ; that $f \in C_{k,\lambda}(G)$, $k \geq 0$, if f has in G k-th derivatives satisfying a Hölder condition of exponent λ; and that $f \in C_{k,\lambda}^{m,\mu}(\Omega)$, $m \geq k$, $m+\mu \geq k+\lambda$, if $f \in C_{k,\lambda}(\Omega)$ and, in addition, f is m times differentiable on Ω and

$$\max_{D^n} \sup_{P \in \Omega} \rho_P^{n-k-\lambda} \left|D^n f(P)\right| < \infty, \qquad k < n \leq m,$$

$$\max_{D^m} \sup_{P,Q \in \Omega} \rho_*^{m+\mu-k-\lambda} \frac{\left|D^m f(P) - D^m f(Q)\right|}{|P-Q|^\mu} < \infty,$$

where D^q is a differentiation operator with respect to x and y of order q; ρ_P is the distance from P to γ; $\rho_* = \min\{\rho_P, \rho_Q\}$; $|P - Q|$ is the lower bound of the lengths of curves joining the points P and Q and lying entirely in Ω.

Construct a grid by the straight lines $x, y = 0, \pm h, \pm 2h, \ldots$. Let Ω_h be the set of grid nodes on Ω such that all interior points of the straight-line segments joining them to the four neighboring nodes belong to Ω; let γ_h be the set of the remaining nodes on Ω;

$$Au(x,y) \equiv (u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h))/4.$$

Introduce the operators l_h^ν and g_h^ν, possessing the following properties. The values of the operators l_h^ν and g_h^ν at each point $P \in \gamma_h$ are expressed linearly in terms of the values of the function at a finite number of points of $\gamma \cup \gamma_h \cup \Omega_h$ at a distance from P not exceeding $\chi_1^\nu h$, and the sum of the moduli of the coefficients of the operator l_h^ν at the values of the function at the points $\gamma_h \cup \Omega_h$ is bounded by the quantity $1 - \chi_2^\nu$, while the sum of the moduli of all coefficients of the operator g_h^ν does not exceed χ_3^ν, where χ_k^ν are positive constants. If the solution of problem (1) satisfies $u \in C_{\nu,0}(\overline{\Omega})$, then

$$u - l_h^\nu u - h^2 g_h^\nu f = g_h^\nu \sum_{k=2}^{\nu^*} \frac{2h^{2k}}{(2k)!}\left(\frac{\partial^{2k} u}{\partial x^{2k}} + \frac{\partial^{2k} u}{\partial y^{2k}}\right) + O(h^\nu),$$

where $\nu^* = [(\nu-1)/2]$, $O(h^\nu)$ uniformly on γ_h. Operators of the type under consideration are constructed for $\nu = 1$ in ([4]), for $\nu = 2$ in ([5]), for $\nu = 3$ in ([6]), for $\nu = 4$ in ([7–9])* and for arbitrary ν in ([10]). In the case $\gamma \in C_{2,0}$, the operators l_h^5 and g_h^5 can be found by the method of undetermined coefficients for $\varkappa_1^5 = 2\sqrt{2}$.

2. Let $u_h = \varphi$ on γ,

$$u_h = Au_h - h^2 f/4 \quad \text{on } \Omega_h, \qquad u_h = l_h^\nu u_h + h^2 g_h^\nu f \quad \text{on } \gamma_h. \tag{2}$$

Denote by c_n, $n = 1, 2, \ldots$, constants independent of ρ and h.

Theorem 1. If $\gamma \in C_{m+2,\lambda}$, $\varphi \in C_{m+2,\lambda}(\gamma)$, $f \in C_{m,\lambda}(\Omega)$, $0 \le m \le 1$, $0 < \lambda < 1$, $\nu \ge 3$, then

$$|u_h - u| \le c_1(\rho + h)^\sigma h^2 \quad \text{on } \Omega_h \cup \gamma_h, \tag{3}$$

where u is the solution of problem (1), u_h is the solution of system (2), $\sigma = \max\{m, \lambda\}$, and ρ is the distance from the node to γ.

Estimate (3), obtained by the method ([11]), refines the uniform estimates of N. S. Bakhvalov ([2]) of order h^2 for $f \equiv 0$. In particular, for $\gamma \in C_{6,\lambda}$, $\varphi \in C_{6,\lambda}(\gamma)$,

$f \in C_{4,\lambda}(\Omega)$, $0 < \lambda < 1$, $\nu = 3$, $\sigma = 1$, estimate (3) is a consequence of the results of ([10]). Denote by $D_h^{q,\tau}$ a linear difference operator with step h, which approximates, on any function continuously differentiable $q+\tau$ times, the differential operator D^q with accuracy $O(h^\tau)$; $\Omega_h^{q,\tau}$ is the subset of the set of nodes in $\overline{\Omega}$ on which the value of the operator $D_h^{q,\tau}$ can be computed from the values of the function on $\Omega_h \cup \gamma_h \cup \gamma$.

Theorem 2. Let $\gamma \in C_{m+2,\lambda}$, $\varphi \in C_{m+2,\lambda}(\gamma)$, $f \in C_{m-\lambda}^{n,\mu}(\Omega)$, $m \geq 0$, $0 < \lambda < 1$, $q \geq 1$, $\tau \geq 1$, $\nu \geq 1$, $\beta = \min\{\tau, 2\}$, $n = \max\{m, q\}$, $\mu = \lambda$ for $q \leq m$, $\mu > \lambda$ for $q = m+1$, $\mu > 0$ for $q > m+1$; then on $\Omega_h^{q,\tau}$

$$|D_h^{q,\tau} u_h - D^q u| \leq c_2 \left(h^\beta + \frac{h^2}{(\rho+h)^{q-m-\lambda}} + \frac{h^\nu}{(\rho+h)^q} \right), \tag{4}$$

where u is the solution of problem (1), u_h is the solution of system (2).

Obviously, for $m \geq q$, $\nu \geq q+2$, $\tau \geq 2$, estimate (4) is uniform on $\Omega_h^{q,\tau}$ of order $O(h^2)$, and, for $\nu - 2 = 2[(m-2)/2] \geq q$, $g_h^\nu = 0$, it follows from ([10]). The right-hand side of inequality (4) for $f \equiv 0$ cannot have higher order in h than the second for any individual solution of problem (1) that is not a polynomial in x, y of degree below the $(q+5)$-th. This assertion is proved by the method ([1]). In the case of the Laplace equation, estimate (3), where $\sigma = 1$, and estimate (4), where $m = 1$, $q = 1$, $\tau \geq 2$, $\nu = 3$, follow, under a stronger requirement on the solution than in Theorems 1 and 2, from the work of V. I. Lebedev ([12]). For $f \equiv 0$, the validity of inequality (4) on the subset $\Omega_h' = \Omega_h^{q,\tau} \cap \Omega'$, where Ω' is a strictly interior subdomain of Ω, with the constant c_2 depending on Ω', follows from the results of V. I. Lebedev ([13]). Bramble and Hubbard ([14]) obtained estimates analogous to those noted, following from ([12,13]), for a more general second-order elliptic equation under the assumption of sufficiently high smoothness of the coefficients of the equation and of the solution.

3. D. F. Davidenko ([15]) proposed an a posteriori estimate of the error of the solution by the grid method for problem (1), using a twice continuously differentiable extension of the grid function to the domain. In ([15]) the properties of the estimate as $h \to 0$ are not clarified. Below, uniform a posteriori error estimates are given, using a local piecewise-polynomial extension of the grid function, with continuity only for the first derivatives, tending to zero as h^2.

* Derivatives of f in the expressions of the operators ([7–9]) are replaced by divided differences.

Denote: $x_i = ih$, $y_j = jh$; R_{ij} is the square $\{x_i < x < x_{i+1},\ y_j < y < y_{j+1}\}$; $\widetilde{\Omega}_h^*$ is the set of vertices of all squares R_{ij} intersecting Ω; Ω_h is the set of nodes from $\widetilde{\Omega}_h$ having the property that all interior points of the line segments joining them with the four nodes located at distance $\sqrt{2}\,h$ belong to Ω; Ω_h^k, $k \geq 1$, is the set of nodes from $\widetilde{\Omega}_h^*$ located at distance h from Ω_h^{k-1}; $r^*(r_*)$ is the radius of the maximal circle whose boundary can be tangent to any point of γ in such a way that this circle has no common points with Ω (lies completely in Ω). Suppose that $0 < h \le r^*/\sqrt{2}$, $\Omega_h^0 \ne \emptyset$. To each point $(x_i, y_j) \in \Omega_h^0$ we assign the polynomial

$$\begin{aligned} P_{ij}(x,y) = (1 &+ \bar{x}_i\delta_x(1 + h^2\delta_y^2/6) + \bar{y}_j\delta_y(1 + h^2\delta_x^2/6) + \\ &+ (\bar{x}_i^2\delta_x^2 + \bar{y}_j^2\delta_y^2)/2 + \bar{x}_i\bar{y}_j\delta_x\delta_y + ((3\bar{x}_i\bar{y}_j^2 - \bar{x}_i^3)\delta_x\delta_y^2 + \\ &+ (3\bar{x}_i^2\bar{y}_j - \bar{y}_j^3)\delta_y\delta_x^2)/6)u_h(x_i, y_j) + \\ &+ (\bar{x}_i(\bar{x}_i^2 - h^2)\delta_x + \bar{y}_j(\bar{y}_j^2 - h^2)\delta_y)f(x_i, y_i)/6, \end{aligned}$$

where $\bar{x}_i = x - x_i$, $\bar{y}_j = y - y_j$, $\delta_x f(x,y) = (f(x+h,y) - f(x-h,y))/2h$, $\delta_x^2 f(x,y) = (f(x-h,y) - 2f(x,y) + f(x+h,y))/h^2$, while δ_y and δ_y^2 are defined analogously. The polynomial P_{ij} corresponding to a point $(x_i, y_j) \in \Omega_h^k$, $k \ge 1$, is set equal to the arithmetic mean of the polynomials corresponding to all points of Ω_h^{k-1} located at distance h from (x_i, y_j); moreover, if $(x_i, y_j) \in \gamma \cup \gamma_h \cup \Omega_h$, then we change the constant term so that $P_{ij}(x_i, y_j) = u_h(x_i, y_j)$. Define the function $\tilde{u}_h$ on $\overline{\Omega}$ as follows:

$$\tilde{u}_h(x,y) = \sum_{\mu=0}^{1}\sum_{\nu=0}^{1} P_{i+\mu,j+\nu}(x,y)\,\psi\left(\frac{\bar{x}_{i+\mu}}{h}\right)\psi\left(\frac{\bar{y}_{j+\nu}}{h}\right), \qquad (x,y) \in \overline{\omega}_{ij},$$

where $\omega_{ij} = \Omega \cap R_{ij}$, $\psi(t) = 1 - 2t^2$ for $|t| \le 1/2$, and $\psi(t) = 2(1 - |t|)^2$ for $|t| \ge 1/2$.

Let $\gamma \in C_{2,0}$; $2d$ be the width of the minimal strip containing Ω; $h_* = r_*/([r_*/h] + 1)$; $\Omega(\rho)$ be the set of points of Ω whose distance from γ is greater than ρ; $\sigma_0 = \Omega(r_*)$; $\sigma_\nu = \Omega(r_* - \nu h_*) \setminus \Omega(r_* - (\nu-1)h_*)$, $\nu \ge 1$;

$$F_\mu = \sup_{(x,y)\in\sigma_\mu} \operatorname{vrai}|f - \Delta\tilde{u}_h|, \qquad F = \sup_{(x,y)\in\Omega} \operatorname{vrai}|f - \Delta\tilde{u}_h|;$$

$k_* = \inf_\gamma k(s)$, where $k(s)$ is the curvature of γ at the current point, with the plus sign at points of convexity; $Z^* = z(r_*)$; $z(t)$ is the solution of the Cauchy problem

$$z''_{tt} + a(t)z'_t = b(t), \qquad 0 \le t \le r_*, \qquad z(0) = z'_t(0) = 0,$$

where $a(t) = 1/(1/k_* - r_* + \mu h_*)$, $b(t) = F_\mu$, $(\mu - 1)h_* \le t < \mu h_*$, which is computed with step h_* in elementary functions.

Theorem 3. If $\gamma \in C_{2,\lambda}$, $\varphi \in C_{2,\lambda}(\gamma)$, $f \in C_\lambda^{(2,\lambda)}(\Omega)$, $0 < \lambda < 1$, $0 < h \le r^*/\sqrt{2}$, $\Omega_h^0 \ne \emptyset$, $\nu \ge 3$, then

$$\max_{\Omega_h \cup \gamma_h} |u_h - u| \leq \max_{\overline{\Omega}} |\tilde{u}_h - u| \leq$$

$$\leq \max_{\gamma} |\tilde{u}_h - \varphi| + Z^* + F_0(d^2 - r_*^2)/2 \leq c_3 h^2, \qquad (5)$$

where u is the solution of problem (1), and u_h is the solution of system (2).

Theorem 4. Let $\gamma \in C_{4,\lambda}$, $\varphi \in C_{4,\lambda}(\gamma)$, $f \in C_{2,\lambda}(\Omega)$, $0 < \lambda < 1$, $0 < h \leq r^*/\sqrt{2}$, $\Omega_h^0 \neq \emptyset$, $\nu \geq 4$; then

$$\max_{\Omega_h \cup \gamma_h} |u_h - u| \leq \max_{\bar{\Omega}} |\tilde{u}_h - u| \leq$$

$$\leq \max_{\gamma} |\tilde{u}_h - \varphi| + Fd^2/2 \leq c_4 h^2, \qquad (6)$$

where u is the solution of problem (1), and u_h is the solution of system (2).

Remarks. 1) Theorems 3 and 4 remain valid if the solution of problem (2) is computed with accuracy $O(h^4)$, and for F_μ one takes the essential maximum of $|f - \Delta\tilde{u}_h|$ over the union of all sets ω_{ij} intersecting σ_μ. 2) In (16) error estimates of other types are proposed, expressed directly in terms of the initial data.

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Note: Figure translations are in progress. See original paper for figures.