ON BICOMPACTA WITH NONCOINCIDING DIMENSIONS

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ON BICOMPACTA WITH NONCOINCIDING DIMENSIONS

ind **and** dim

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At present there is a large number of examples of bicompacta with noncoinciding dimensions ind and dim (see (1-4)). In the present note a number of examples of this sort are set forth. In some respects they are better than the preceding ones.

§ 1*. **A bicompactum** P **with** $\dim P = 1$**,** $\operatorname{ind} P = \operatorname{Ind} P = 2$**.** The ordinary square $I^2 = I \times I$, where $I = [0, 1]$, carries the Euclidean structure (under which the factors are orthogonal and have length 1), and therefore we can measure the magnitudes of angles. By a metric we shall mean the Euclidean metric.

In the product $P = I^2 \times S$, where S is the ordinary circle, we introduce a somewhat nonstandard topology. Let $I_0 \subseteq I$ be the set of rational points of the interval, and $I_0^2 \subseteq I^2$ the corresponding set of points of the square both of whose coordinates are rational. We shall identify points of the circle S with directions of rays emanating from a point, taking the direction of the axis of abscissae to be the zero direction. Let V be a set open on the circle $\{x\} \times S$. By the symbol V^* we denote: 1) the union of all rays lying in I^2 (without the point x) which emanate from the point x at angles corresponding to the points of the set V, if $x \in I_0^2$; 2) the union of all circles lying in I^2 (or, more precisely, of their intersections with I^2) with center at the point x and with radii r such that $1/r$ measures an angle lying in V, if $x \in I^2 \setminus I_0^2$.

Let $\pi : P \to I^2$ be the projection onto the factor, and let U be a neighborhood of the point x in I^2. As a neighborhood of any point of a set $V \subseteq \{x\} \times S$ we declare the set $V \cup \pi^{-1}(V^* \cap U)$. As is easy to see, the topology is given correctly, and the constructed space P is a bicompactum.

The estimate $\dim P \geq 1$ is obvious. The estimate $\dim P \leq 1$ is obtained as follows. Take an arbitrary covering ω of the space P. Contract to points those sets of the form $\{x\} \times S$, where $x \in I^2 \setminus I_0^2$, which are completely covered by at least one element of the covering (as is easy to see, all such sets will be covered, except perhaps for a finite number). This mapping will be an ω-mapping of the bicompactum P onto a metrizable compactum whose one-dimensionality is obvious. Thus $\dim P \leq 1$.

Let $V \subset P$ be any open set whose complement has nonempty interior. As is easy to see, in the boundary of the set $\operatorname{Int} \pi(V)$ there will lie some connected nondegenerate set F. Since such a set cannot be countable, there will be found a point $x \in F \setminus I_0^2$. Then, from the connectedness of the set F, the definition of the topology in P, and the closedness of the boundary of the set V, we obtain that the one-dimensional set $\{x\} \times S$ lies entirely in the boundary of the set V. From what has been said it follows that $\operatorname{Ind} P \geq \operatorname{ind} P \geq 2$. The estimate $\operatorname{Ind} P \leq 2$ follows from obvious geometric considerations.

§ 2. **A bicompactum P_i with $\dim P_i = 1$, $\operatorname{ind} P_i = \operatorname{Ind} P_i = i$.** The bicompactum P was obtained from the square by inserting, in place of points, circles with "expansion" of neighborhoods of a point "by sectors" –case 1)–or "by rings–

* The construction of §§ 1-4 is usefully compared with the example of V. V. Fedorchuk.

–for the case of circles, case 2). But instead of circles in the second case one may insert (below we shall show how) an already constructed bicompactum. As before, every boundary will contain entirely some copy of the space being inserted, which raises the inductive dimensions by at least one; whereas dim, as before, will be equal to 1. However, precise upper estimates of the inductive dimensions are very difficult to obtain. Therefore we shall slightly modify the geometry of the example.

As P_1 take the square in which all points are replaced by circles with convergence by sectors. Suppose we have already constructed in this way a bicompactum P_{i-1} with $\dim P_{i-1} = 1$, $\operatorname{ind} P_{i-1} = \operatorname{Ind} P_{i-1} = i-1$. Let $\pi_{i-1} : P_{i-1} \to I^2$ be the natural projection onto the square. Let $f : (0,1] \to I^2$ be at most a two-to-one mapping of the half-interval $(0,1]$ into the square I^2, which is a linear parametrization of some polygonal line such that $[f((0,r))] = I^2$, if $0 < r \leqslant 1$.

Replace the point $x = (r_1, r_2)$ of the square I^2 by a circle with convergence by sectors if $r_1 \neq 0$, and by the bicompactum P_{i-1} if $r_1 = 0$. In the latter case neighborhoods are defined as follows. Let V be an open set in this copy of the bicompactum P_{i-1}, $V^* = \operatorname{Int} \pi_{i-1}(V)$, and let V^{**} be the union of all circles with center at the point x and radii from $f^{-1}(V^*)$; let U be a neighborhood of the point x in I^2, and $\pi_i : P_i \to I^2$ the projection of the set P_i, obtained as a result of the insertions, onto the square, gluing the inserted sets to points. As a neighborhood of any point of V we call the set $V \cup \pi_i^{-1}(V^{**} \cap U)$.

The estimates $\dim P_i = 1$, $\operatorname{Ind} P_i \geqslant \operatorname{ind} P_i \geqslant i$ are obtained almost in the same way as for the bicompactum P. We shall show that $\operatorname{ind} P_i \leqslant i$. For points lying in the sets $\pi_i^{-1}((r_1, r_2))$, where $r_1 \neq 0$, one can, obviously, find neighborhoods with zero-dimensional boundaries. Let $\pi_i(y) = (0, r_2) = x$, let

V_1 be a neighborhood of the point y in the subspace $\pi_i^{-1}(x)$ with an $(i-2)$-dimensional boundary, and let $V = V_1 \cup \pi_i^{-1}(V_1^{**} \cap U)$ be the neighborhood of the point y in the space P_i constructed from it. Let y' be a boundary point of the set V_1. We can find an arbitrarily small neighborhood V_2 of the point y' in the subspace $\pi_i^{-1}(x)$, which is the union of a countable number of open subsets G_n, $n = 1, 2, \dots$, $[G_n] \subset G_{n+1}$. Let V_2^{**} and $[G_n]^{**}$ be the sets constructed from V_2 and $[G_n]$, respectively, by the method described above. These sets consist of concentric annuli. Let L be an annulus which is a component of the open set V_2^{**}, whose inner radius is greater than $1/n$. Let L^* be an annulus, $[G_n]^{**} \cap L \subset L^* \subset L$, whose boundary does not meet the boundary of the set V_1^{**}; such an annulus, as is easy to see, exists. Let Λ be the union of such annuli L^* over all components L of the set V_2^{**}. As is easy to see, the set $V_2 \cup \Lambda$ is open in P_{i+1}, and the intersection of its boundary with the boundary of the set V lies in the boundary of the set $V_1 \subset \pi_i^{-1}(x)$, and consequently has dimension $\leqslant i-2$. For the remaining points of the boundary of the set V, arbitrarily small neighborhoods with boundaries of dimension $\leqslant i-2$ are found in an obvious way. Thus ind of the boundary of the set V is $i-1$, and consequently $\operatorname{ind} P_i \leqslant i$. The inequality $\operatorname{Ind} P_i \leqslant i$ is proved analogously.

§ 3. **A perfectly normal bicompactum Q with** $\dim Q = 1$, $\operatorname{ind} Q = \operatorname{Ind} Q = 2$**.** We shall construct this bicompactum under the assumption that $2^{\aleph_0} = \aleph_1$.

We shall show that, if $2^{\aleph_0} = \aleph_1$, then in the square I^2 there exists a set L with the following properties: a) the intersection of L with any zero-dimensional compactum is at most countable; b) every connected compactum contains at least one point of the set L; c) $L \cap I_0^2 = \emptyset$.

For this, enumerate by transfinite ordinals less than ω_1 all zero-dimensional compacta lying in I^2; enumerate by transfinite ordinals less than ω_1 all connected compacta lying in I^2. As is easy to see, every zero-dimensional compactum is nowhere dense in any connected compactum containing it, so that, if from a connected compactum with number $\alpha < \omega_1$ we discard the points

of all zero-dimensional compacta with smaller indices and the points of the set I_0^2, the remaining set will be nonempty. Choose in it an arbitrary point and denote it by x_α. Obviously the set $L = \{x_\alpha\}_{\alpha<\omega_1}$ satisfies the stated conditions.

As is easy to see, as a result of decomposing the space P from § 1, whose non-singleton elements are the sets $\{x\} \times S$, where $x \in I^2 \setminus (I_0^2 \cup L)$, one obtains a (Hausdorff) compactum Q. Let $\pi' : Q \to I^2$ be the mapping induced by the projection π.

We shall show that the compactum Q is perfectly normal. For this it is enough to show that into any family W of open subsets of Q one can inscribe no more than a countable family with the same union. Let $\{u_1, u_2, \dots\}$ be the set of those elements of some countable base of the square I^2 whose complete preimages under π' are inscribed in elements of the family W. As is easy to see, the sets $\{\pi'^{-1}(u_1), \pi'^{-1}(u_2), \dots\}$ fail to cover only the set M of those points of the space Q, covered by the family W, which the mapping π' sends into the boundary of the set $\bigcup_{i=1}^{\infty} u_i$. We shall show that the set $M_0 = \pi'(M)$ is at most countable. For each point $m \in M_0 \setminus I_0^2 \subset L$ choose some point $x(m) \in \pi'^{-1}(m)$ covered by the family W. In some element of the family W there lies a neighborhood of the point $x(m)$ of the form

$$V \cup (\pi'^{-1}(V^* \cap U))$$

(see the construction of the space P), where U is an ε-neighborhood of the point m. Let H_n be the set of those points $m \in M_0$ for which this $\varepsilon > 1/n$. It is not hard to verify that the set $[H_n]$ is zero-dimensional, and therefore $H_n \subset L$ is at most countable, so that the set

$$M_0 = \bigcup_{n=1}^{\infty} H_n \cup (M_0 \cap I_0^2)$$

is also countable. For each point $m \in M_0$ it is not hard to find a countable family $V(m)$ of open subsets of Q, inscribed in W, which covers in $\pi'^{-1}(m)$ everything that is covered by the family W. Then the family

$$\{\pi'^{-1}(u_1), \pi'^{-1}(u_2), ...\} \cup \bigcup_{m \in M_0} V(m),$$

as is easy to see, will be countable, will be inscribed in the family W, and will have the same union as W. Thus we have proved that the compactum Q is perfectly normal.

The estimates of the dimensions of the compactum Q are obtained in the same way as in the preceding cases.

§ 4. **A perfectly normal compactum Q_i with $\dim Q_i = 1$, $\operatorname{ind} Q_i = \operatorname{Ind} Q_i = i$.** These compacta are constructed under the assumption that $2^{\aleph_0} = \aleph_1$.

The compactum $Q_2 = Q$ has already been constructed by us. The induction is carried out in the same way as in § 2. Suppose that we have already constructed the compactum Q_{i-1}, $\pi'_{i-1} : Q_{i-1} \to I^2$, a mapping of this compactum onto the square. Let $f' : (0, 1] \to I^2$ be an at most two-to-one mapping of the half-interval $(0, 1]$ into the square I^2, which is a linear parametrization of some broken line, such that the $1/2n$-neighborhood of the set

$$f'((1/(n + 1), 1/n))$$

covers I^2. The difference from the corresponding place in § 2 included here is necessary for the perfect normality of the compactum that will be constructed by us.

We do not replace the point $x \in I^2$ by anything if $x \in I^2 \setminus (I_0^2 \cup L)$; we replace it by a circle with convergence by sectors if $x \in I_0^2$; and we replace it by the compactum Q_{i-1} if $x \in L$. In the last case define neighborhoods as follows.

Let V be a subset open in this copy of the compactum Q_{i-1}, $V^* = \operatorname{Int} \pi'_{i-1}(V)$, V^{**} the union of all neighborhoods with center at the point x and radii from ${f'}^{-1}(V^*)$, U a neighborhood of the point x in I^2, and $\pi'_i : Q_i \to I^2$ the projection of the set Q_i onto the square resulting from the insertions, gluing the inserted sets to the points. As a neighborhood of any point of V we shall call the set

$$V \cup \pi_{i'}^{-1}(V^{**} \cap U).$$

The proof of the perfect normality of the compactum is carried out in the same way as in § 3.

The estimates of the dimensions are obtained in the same way as before.

§ 5. **Remarks.** As is easy to see, the series of examples constructed in §§ 2 and 4 can be extended to transfinite dimensions.

Taking the free sum of the required number of bicompacts and compactifying this locally bicompact space by one point, one can construct bicompacts P_∞ and Q_∞ with $\dim P_\infty = 1$, $\operatorname{ind} P_\infty = \infty$, $\dim Q_\infty = 1$, $\operatorname{ind} Q_\infty = \infty$. Let $m \geq n$, $P_{m,n} = P_m \cup I^n$, $Q_{m,n} = Q_m \cup I^n$. As is easy to see, $\dim P_{m,n} = n$, $\operatorname{ind} P_{m,n} = m$, $\dim Q_{m,n} = n$, $\operatorname{ind} Q_{m,n} = m$.

§ 6. **Construction, from a bicompact R with $\dim R = l$, $\operatorname{ind} R = m$, $\operatorname{Ind} R = n$, of a bicompact R^* with $\dim R^* = l$, $\operatorname{ind} R^* = m + 1$, $\operatorname{Ind} R^* = n + 1$.**

On the set $C_0 = I \times I \times D$, where $D = \{0, 1\}$, introduce the lexicographic order as follows: $(r_1, r_2, r_3) > (r'_1, r'_2, r'_3)$ if $r_1 > r'_1$, or $r_1 = r'_1$, $r_2 > r'_2$, or $r_1 = r'_1$, $r_2 = r'_2$, $r_3 > r'_3$.

Let $\varphi : \theta \to R$ be a continuous mapping of some zero-dimensional bicompact θ onto our bicompact R. At least one such bicompact θ exists: every bicompact is the image of some closed subset of the bicompact D^τ.

On the set $C = C_0 \cup (I \times I \times \theta)$, introduce a topology in the following way. The sets $\{r_1\} \times \{r_2\} \times \theta$ shall be regarded as open-and-closed subspaces with the topology of θ. Convergence to points of C_0 is interval convergence; moreover it is assumed that the set $\{r_1\} \times \{r_2\} \times \theta$ lies between the points $(r_1, r_2, 0)$ and $(r_1, r_2, 1)$.

Let z be a point not belonging to R. On the set $I^* = I \times (\{z\} \cup R)$ introduce a topology in the following way. The sets $\{r\} \times R$, $r \in I$, shall be regarded as open-and-closed subspaces with the topology of R. A standard neighborhood of a point $r \in I \times \{z\}$ has the form $(l \times (\{z\} \cup R)) \setminus (\{r\} \times R)$, where l is an interval in I.

In the Tikhonov product $R_1 = I^* \times C$ make the following identifications. In the set $\{(r, z)\} \times (\{r_1\} \times \{r\} \times \theta)$ make the identification φ. Let R^* be the bicompact obtained as a result of these identifications.

The estimates $\dim R^* = l$, $\operatorname{ind} R^* \leq m+1$, $\operatorname{Ind} R^* \leq n+1$ are obvious. By standard arguments, an example of which can be found in (4), it is proved that every partition between the points $(0,z)\times(0,0,1)$ and $(1,z)\times(0,0,1)$ contains a copy of the space R, and, consequently, $\operatorname{ind} R^* \geq m+1$, $\operatorname{Ind} R^* \geq n+1$.

In order that the first axiom of countability hold in the bicompact R^*, it is necessary and sufficient that it hold in the bicompacts R and θ.

§ 7. **A bicompact S_i with $\dim S_i = 1$, $\operatorname{ind} S_i = \operatorname{Ind} S_i = i$, $S_i = \bigcup_{j=1}^{i} S_i^j$, where S_i^j is a bicompact that is one-dimensional in all senses.**

Suppose that we have already constructed the bicompact S_{i-1}. Carrying out the construction of § 6, we obtain the bicompact S_i. The representation of this bicompact as a union of a finite number of bicompacts that are one-dimensional in all senses is obvious.

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REFERENCES CITED

1. O. V. Lokutsievskii, DAN, 67, No. 2, 217 (1949).
2. A. L. Lunts, DAN, 66, No. 5, 801 (1949).
3. V. V. Fedorchuk, DAN, 182, No. 2, 275 (1968).
4. V. V. Filippov, DAN, 186, No. 5, 1020 (1969).

Note: Figure translations are in progress. See original paper for figures.