SOLUTION OF THE DIRICHLET PROBLEM FOR A HALF-SPACE

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Abstract

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SOLUTION OF THE DIRICHLET PROBLEM FOR A HALF-SPACE

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In the present note we state theorems which are, in a certain sense, analogues of Fatou’ s theorem for the Poisson integral in space, and on the basis of these theorems we solve the Dirichlet problem for a half-space in the formulation of N. N. Luzin ((*1*), p. 85). This problem for a ball was considered in (*2*).

1. Notation and definitions. In this paper the following notation is adopted: $R = (-\infty < x < \infty;\ -\infty < y < \infty)$; $\omega(P; h)$–the circle of radius h with center at the point $P \in R$; $\widetilde{L}(R)$–the set of functions $f(x, y)$ such that the function $f(x, y)/(1 + x^2 + y^2)^{3/2}$ is integrable on R; Z^+–the set of points (x, y, z) for which $z > 0$; $\Delta_{xy} = \partial^2/\partial x^2 + \partial^2/\partial y^2$; $U(f; x, y, z)$–the Poisson integral for the function $f(x, y)$ in the space Z^+, i.e.

$$U(f; x, y, z) = \frac{z}{2\pi} \iint_R \frac{f(t, \tau)\, dt\, d\tau}{[(t-x)^2 + (\tau - y)^2 + z^2]^{3/2}};$$

the symbol $(h, k)_\lambda \to 0$ means that $h \to 0$, $k \to 0$, and

$$1/\lambda \le |h/k| \le \lambda, \quad \lambda \ge 1; \qquad M(x, y, z) \xrightarrow{\Lambda} P(x_0, y_0, 0)$$

means that the point M tends to P along paths non-tangential to R, i.e. there exists a positive number k such that

$$z/\sqrt{(x-x_0)^2 + (y-y_0)^2} \ge k.$$

Definitions. 1) The generalized Laplace operator is defined by the formula (*3*, *4*):

$$\Delta^* f(P) = \lim_{h\to 0} \frac{8}{\pi h^4} \iint_{\omega(P;h)} [f(Q) - f(P)]\, d\omega_Q;$$

2) The derivatives $Df(x,y)$, $\widetilde{D}f(x,y)$, $D_\lambda f(x,y)$, $C_1Df(x,y)$, $C_1D^*f(x,y)$, and $C_{1\lambda}Df(x,y)$ of the function $f(x,y)$ at the point $P(x,y)$ are defined as follows:

$$Df(x,y) = \lim_{h,k\to 0} \frac{f(x+h,y+k) - f(x,y+k) - f(x+h,y) + f(x,y)}{hk};$$

$$\widetilde{D}f(x,y) =$$

$$\lim_{h,k\to 0} \frac{f(x+h,y+k) + f(x-h,y+k) + f(x+h,y-k) + f(x-h,y-k) - 4f(x,y)}{h^2+k^2};$$

$$D_\lambda f(x,y) = \lim_{(h,k)_\lambda\to 0} \frac{f(x+h,y+k) - f(x,y+k) - f(x+h,y) + f(x,y)}{hk};$$

$$C_1Df(x,y) = \lim_{h,k\to 0} \frac{4}{h^2k^2} \int_x^{x+h} \int_y^{y+k} [f(t,\tau) - f(t,y) - f(x,\tau) + f(x,y)]\, dt\, d\tau;$$

$$C_1D^*f(x,y) = \lim_{h,k\to 0} \frac{1}{h^2k^2} \int_0^h \int_0^k [f(x+t,y+\tau) - f(x-t,y+\tau) - \\ - f(x+t,y-\tau) + f(x-t,y-\tau)]\, dt\, d\tau;$$

$$C_{1\Lambda}Df(x,y) = \lim_{(h,k)\xrightarrow{\Lambda} 0} \frac{4}{h^2k^2} \int_x^{x+h} \int_y^{y+k} [f(t,\tau) - f(t,y) - f(x,\tau) + f(x,y)]\, dt\, d\tau.$$

2. Fatou theorems for the Poisson integral in the space Z^+

The Fatou theorem for the Poisson integral in the space Z^+ has certain special features. The behavior of the differentiated Poisson integral depends essentially on the sense in which the density of the integral is differentiated.

The following theorems hold:

Theorem 1. a) If $f(x,y) \in \widetilde{L}(R)$ and at the point $P(x,y,0)$ there exists a finite derivative $\Delta^* f(x,y)$, then

$$\lim_{z\to 0+} \Delta_{xy} U(f;x,y,z) = \Delta^* f(x,y).$$

b) There exists a continuous function $f(x,y)$ such that $\widetilde{D}f(0,0)=0$, but

$$\lim_{\substack{(x,y,z)\to(0,0,0)\\ \Lambda}} \Delta_{xy} U(f;x,y,z)$$

does not exist.

Theorem 2. If $f(x,y) \in \tilde{L}(R)$ and at the point $P(x,y,0)$ there exists a finite derivative $C_1 D^* f(x,y)$, then

$$\lim_{z\to 0+} \frac{\partial^2 U(f;x,y,z)}{\partial x\,\partial y} = C_1 D^* f(x,y).$$

Theorem 3. If $f(x,y) \in \tilde{L}(R)$ and at the point $P(x_0,y_0,0)$ there exists a finite derivative $C_1 Df(x_0,y_0)$, then

$$\lim_{\substack{(x,y,z)\to(x_0,y_0,0)\\ \Lambda}} \frac{\partial^2 U(f;x,y,z)}{\partial x\,\partial y} = C_1 Df(x_0,y_0).$$

Theorem 4. There exists a continuous function $f(x,y)$ such that $D_\Lambda f(0,0)=0$, but

$$\lim_{z\to 0+} \frac{\partial^2 U(f;0,0,z)}{\partial x\,\partial y} = +\infty.$$

Theorem 5. There exists a continuous function $f(x,y)$ such that $C_{1\Lambda} Df(0,0)=0$, but

$$\lim_{z\to 0+} \frac{\partial^2 U(f;0,0,z)}{\partial x\,\partial y} = +\infty.$$

Theorem 6. a) If $f(x,y) \in \tilde{L}(R)$ and $f(x,y)$, at the point $P(x_0,y_0,0)$, has a total differential $df(x_0,y_0)$, then

$$\lim_{\substack{(x,y,z)\to(x_0,y_0,0)\\ \Lambda}} \frac{\partial U(f;x,y,z)}{\partial x} = \frac{\partial f(x_0,y_0)}{\partial x};$$

$$\lim_{\substack{(x,y,z)\to(x_0,y_0,0)\\ \Lambda}} \frac{\partial U(f;x,y,z)}{\partial y} = \frac{\partial f(x_0,y_0)}{\partial y}.$$

b) There exists a continuous function $f(x,y)$ such that $\partial f(0,0)/\partial x = \partial f(0,0)/\partial y = 0$, but

$$\lim_{z \to 0+} \frac{\partial U(f; 0, 0, z)}{\partial x} = +\infty.$$

3. The Dirichlet Problem for a Half-Space

The Dirichlet problem for the Laplace equation consists in determining a function $U(x, y, z)$ in the domain Z^+ with boundary R, satisfying the equation $\Delta U = 0$ and the boundary condition $U|_R = f(x, y)$. In paper [5], a solution of this problem is given in the case when $f(x, y) \in \tilde{L}(R)$. In the present note this problem is solved for the case when the boundary function is measurable and finite almost everywhere on R, i.e., in the formulation of N. N. Luzin.

Theorem 7. Let $f(x, y)$ be an arbitrary measurable function, finite almost everywhere on R. Then there exists a bounded continuous function $F(x, y)$ such that almost everywhere on R

$$DF(x, y) = f(x, y).$$

Theorem 8. Let $f(x, y)$ be an arbitrary measurable function, finite almost everywhere on R. Then there exists a harmonic function $U(x, y, z)$ in Z^+ such that

$$\lim_{(x,y,z) \to (x_0, y_0, 0)} U(x, y, z) = f(x_0, y_0)$$

almost everywhere on R.

All the theorems stated above are valid for any n-dimensional Euclidean space.

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Note: Figure translations are in progress. See original paper for figures.