ON COMPACTIFICATIONS ONTO COMPACTA

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 513.831

MATHEMATICS

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(Presented by Academician P. S. Aleksandrov on 12 I 1970)

The following definition is known (1).

Definition 1. Let X be a topological space. A point $x \in X$ is called a **point of noncompactness** of the space X if at the point x there exists no base consisting of open sets with bicompact closures.

The set N of all points of noncompactness of the space X is, obviously, closed in X.

Theorem 1. *A metric space with a countable base is compactifiable onto a complete metric space if and only if the set of all its points of noncompactness is compactifiable onto some complete metric space.*

Unfortunately, the analogous theorem on compactifications onto compacta is false.

Example 1. The space X is a subset of the plane R^2:

$$X = \bigcup_{n=1}^{\infty} [0,1] \times \left(\frac{1}{n}\right) \cup (0,0) \cup (1,0).$$

As is known (1), the space X is not compactifiable onto any compactum, even though the set of all points of noncompactness N of the space X is a compactum:

$$N = \{(0,0); (1,0)\}.$$

Example 2. The space Y is a subset of the space R^3:

$$Y = ([0,1] \times [0,1] \times (0,1]) \cup X,$$

where X is the space constructed above. The set of all points of noncompactness of the space Y coincides exactly with X, and consequently is not compactifiable onto any compactum. Nevertheless, one can show that the space Y is compactifiable onto some compactum Z. (The proof of this fact almost repeats the proof of the theorem on the "burnt" polyhedron (^{1}).)

Theorem 2. *Let X be a Hausdorff space and let $A, B \subseteq X$ be closed subsets of the space X such that:*

1) *A and B are compactifiable onto bicompacta;*

2) *$X = A \cup B$;*

3) *$K = A \cap B$ is bicompact.*

Then the space X is compactifiable onto a bicompactum.

Proof. Let $f_1 : A \to A'$ and $f_2 : B \to B'$ be compactifications of the spaces A and B onto bicompacta A' and B', respectively. We construct a bicompactum X' onto which the space X is compactified. The underlying set of the space X' coincides with the set X. The topology of the space X' is arranged as follows: if a point $x \in A \setminus K \subseteq X'$, a base at the point x is formed by all sets of the form $f_1^{-1}(V)$, where V is an open set in $A' \setminus f(K)$ containing the point $f_1(x)$. A base at points $x \in B \setminus K \subseteq X'$ is defined analogously. Let a point $x \in K \subset X'$. A base at the point x is formed by all sets of the form $f_1^{-1}(U) \cup f_2^{-1}(V)$, where U and V are neighborhoods of the point x in the spaces A' and B', respectively, such that $f_1^{-1}(U) \cap K = f_2^{-1}(V) \cap K$. It is not difficult to verify that the space X' is Hausdorff, and the spaces A' and B' can be

identify with the subspaces A and B of the space X'. Hence it follows that X' is bicompact. It is clear that the space X is compactified onto the space X'. The theorem is proved.

Theorem 3. *Let X be such a completely regular space that: 1) the set N of all points of noncompactness of the space X is finite—$N = \{x_i\}$, $i = 1, 2, \dots, n$. 2) For each point $x_i \in N$ there exists a neighborhood Ox_i with bicompact boundary such that $Ox_i \cap Ox_j = \emptyset$ for any points $x_i, x_j \in N$, $i \neq j$. Then the space X is compactified onto a bicompactum.*

Proof. From the conditions of the theorem there follows the existence of such neighborhoods Γ_i of the points $x_i \in N$, $i = 1, 2, \dots, n$, that $x_i \in \Gamma_i \subset [\Gamma_i]_X \subset Ox_i$, and the set Γ_i has bicompact boundary for all i. Consider βX, the maximal bicompact extension of the space X. As is known $^{(2)}$, βX is a perfect extension of the space X; therefore

$$\mathrm{Fr}_{\beta X}\, O\langle \Gamma_i \rangle = [\mathrm{Fr}_X\, \Gamma_i]_{\beta X} = \mathrm{Fr}_X\, \Gamma_i, \quad i = 1, 2, \dots, n,$$

where $O\langle\ \rangle$ is the operator of maximal cut-out in βX. Let

$$A_i = [\Gamma_i]_{\beta X} \setminus (X \setminus N).$$

These are, obviously, closed subsets of βX, and for them we have $A_i \cap X = x_i$ and

$$A_i \cap A_j \subset \operatorname{Fr}_{\beta X} O\langle \Gamma_i \rangle \cap \operatorname{Fr}_{\beta X} O\langle \Gamma_j \rangle \subset [\Gamma_i]_X \cap [\Gamma_j]_X = \emptyset,$$

if $i \neq j$. Put

$$A_0 = (\beta X \setminus X) \setminus \bigcup_{i=1}^{n} A_i.$$

We shall show that A_0 is closed in βX. Indeed, since

$$[\beta X \setminus X]_{\beta X} \cap X = N,$$

we have

$$[A_0]_{\beta X} \cap X = \emptyset.$$

Further, it is easy to see that

$$[A_0]_{\beta X} \cap \left(\bigcup_{i=1}^{n} A_i \right) = [A_0]_{\beta X} \cap \left(\bigcup_{i=1}^{n} \operatorname{Fr}_{\beta X} O\langle \Gamma_i \rangle \right) = [A_0]_{\beta X} \cap X = \emptyset.$$

Take an arbitrary point $y \in X$, $y \notin N$, and consider the continuous decomposition of the space βX whose elements are: the points of the set $X \setminus (N \cup y)$, the bicompacts A_i, $i = 1, 2, \dots, n$, and the bicompact $A_0 \cup y$. The space of this continuous decomposition is some bicompactum Y, and the restriction of the natural mapping $f : \beta X \to Y$ to the space X is a compactification of this space onto the bicompactum Y: $f|_X : X \to Y$. The theorem is proved.

In what follows we shall need the following lemma:

Lemma 1. *Let X be a metric space with a countable base such that: 1) X is an absolute F_σ, and 2) $\dim X = 0$, and let ρ be some metric of the space X. Then X can be represented in the form*

$$X = \bigcup_{i=1}^{\infty} K_i,$$

where the K_i are pairwise disjoint compacta and $\operatorname{diam} K_i \to 0$ as $i \to \infty$, where $\operatorname{diam} K_i$ is considered in the metric ρ.

From Lemma 2 the following easily follows.

Theorem 4. *Let X be a metric space with a countable base. In order that the space X have a compact extension with a countable remainder, it is necessary and sufficient that X be π-compact and be an absolute G_δ.*

Corollary (3)**.** *Every π-compact metric space with a countable base that is an absolute G_δ is compactified onto a compactum.*

Definition 2. A metric space X with a countable base is called **weakly π-compact** if there exists a system of sets open in the space X,

$$\{U_i\}, \quad i = 1, 2, \dots,$$

with compact boundaries, such that for any points $x, y \in X$, $x \neq y$, one can find open sets

$$U, V \in \{U_i\},$$

for which

$$x \in U, \quad y \in V, \quad \text{and} \quad U \cap V = \emptyset.$$

Theorem 5. *Let X be a weakly π-compact space. Then the space X is compactified onto some π-compact space Y, which is metrizable and has a countable base.*

Proof. Since the space X is weakly π-compact, there exists such a system $\mathcal{U}$ of open subsets of X:

$$\mathcal{U} = \{U_i\}, \quad i = 1, 2, \dots,$$

with compact boundaries, that for any points $x, y \in X$, $x \neq y$, one can find such $U, V \in \mathcal{U}$, for which

$$x \in U, \quad y \in V, \quad \text{and} \quad U \cap V = \emptyset.$$

A certain system of subsets $\mathfrak{A} = \{A_\alpha\}$ of the space X is called **algebraically closed** if it satisfies the following

conditions: 1) if A_{α_0} and $A_{\alpha_1} \in \mathfrak{A}$, then also $A_{\alpha_0} \cap A_{\alpha_1} \in \mathfrak{A}$; 2) if $A_{\alpha_0} \in \mathfrak{A}$, then also $X \setminus [A_{\alpha_0}]_X \in \mathfrak{A}$. Using induction, it is not difficult to prove that there exists a minimal algebraically closed system $\overline{\mathcal{U}}$ containing the system $\mathcal{U}$ and having the following properties: a) $\overline{\mathcal{U}}$ consists of a countable number of open sets, $\overline{\mathcal{U}} = \{V_j\}$, $j = 1, 2, \dots$; b) every $V_j \in \overline{\mathcal{U}}$ has compact boundary. Construct the space Y. The underlying set of the space Y is the set X, and a base of open sets is formed by the elements of the system $\overline{\mathcal{U}}$. It is clear that Y is a Hausdorff space with a countable base. Let us show that the space Y is π-compact. Let $y \in Y$, and let Oy be an arbitrary neighborhood of the point y. There exists an open set U of the space Y, $U \in \overline{\mathcal{U}}$, such that $y \in U \subseteq Oy$. Since the set $X \setminus [U]_X \in \overline{\mathcal{U}}$, the set $[U]_X$ is closed in the space Y and $[U]_X = [U]_Y$; therefore $\operatorname{Fr}_Y U = \operatorname{Fr}_X U$ and is a compactum. Let us prove regularity, and consequently metrizability, of the space Y. Let $y \in Y$, and let Oy be an arbitrary neighborhood of the point y. Without loss of generality one may assume that $Oy \in \overline{\mathcal{U}}$. If $[Oy]_Y = Oy$, then the assertion is obvious. Otherwise, for every point $x \in \operatorname{Fr}_Y Oy$, by the Hausdorffness of the space Y, there exists a neighborhood Ox in the space Y such that $[Ox]_Y \not\ni y$. The set of all such neighborhoods forms an open cover of the compactum $\operatorname{Fr}_Y Oy$; therefore there is a finite set of points $x_i \in \operatorname{Fr}_Y Oy$ such that $\operatorname{Fr}_Y Oy \subseteq \bigcup_i Ox_i$ and $[\bigcup_i Ox_i]_Y \not\ni y$. Let

$$U = Oy \cap (Y \setminus [\bigcup_i Ox_i]_Y).$$

It is not difficult to see that $y \in U \subseteq [U]_Y \subseteq Ou$. Since the space X is condensed onto the space Y, the theorem is proved.

Definition 3. Let X be a topological space and let $x \in X$. The point x is called a **point of non-π-compactness** of the space X if there is no base at the point x in the space X consisting of open sets with bicompact boundaries. If the space X is metric, the set M of all points of non-π-compactness of the space X is a set of type F_σ.

Theorem 6. Let X be a weakly π-compact complete metric space with a countable base, the set of all points of non-π-compactness M of which is a compactum. Then the space X is condensed onto some compactum.

Proof. Since the space X is weakly π-compact, there exists a system $\{U_i\}$, $i = 1, 2, ...$, of open sets with compact boundaries such that, for any points $x, y \in X$, one can find $U, V \in \{U_i\}$ for which $x \in U$, $y \in V$, and $U \cap V = \emptyset$. The set M of points of non-π-compactness of X is a compactum; therefore the set $X \setminus M$ is open in the space X, and every point $x \in X \setminus M$ has, in the space X, a base consisting of open sets with compact boundaries. Consequently, there exists a system $\{V_j\}$, $j = 1, 2, ...$, of open sets of the space X with compact boundaries such that, for every point $x \in X \setminus M$, the set of all elements of this system containing the point x forms a base at the point x in the space X. Let $\mathcal{U} = \{U_i\} \cup \{V_j\}$, $i, j = 1, 2, ...$ As in the proof of Theorem 5, construct a π-compact metric space Y, using the system $\mathcal{U}$. If $f : X \to Y$ is the condensation obtained in the proof of the preceding theorem, then it is not difficult to verify that $f|_{X \setminus M} : X \setminus M \to f(X \setminus M) \subseteq Y$ is a homeomorphism. Since $f|_M : M \to f(M)$ is also a homeomorphism, the space Y, as the union of two absolute G_δ' s—$f(X \setminus M)$ and $f(M)$—is an absolute G_δ, and, by the corollary to Theorem 4, is condensed onto some compactum Z. Therefore the space X is also condensed onto the compactum Z. The theorem is proved.

The following theorem is proved analogously:

Theorem 7. Let X be a complete metric space with a countable base, the set of all points of non-π-compactness M of which is a discrete and closed subset of X. Let $M = \{x_i\}$, $i = 1, 2, ...$ Suppose

that there exists a disjoint system $\{Ox_i\}$ of open sets with compact boundaries such that $x_i \in Ox_i$, $i = 1, 2, ...$ Then the space X is compactified on a certain compactum Z.

Corollary. Let X be a complete metric space with a countable base. If the space X has only one point of non-π-compactness, then it is compactified on a certain compactum.

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Received
8 I 1970

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Note: Figure translations are in progress. See original paper for figures.

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