ON THE UNIQUENESS OF THE SOLUTION OF THE CAUCHY PROBLEM FOR GENERAL LINEAR EQUATIONS

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Abstract

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ON THE UNIQUENESS OF THE SOLUTION OF THE CAUCHY PROBLEM FOR GENERAL LINEAR EQUATIONS

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The question of uniqueness classes for the solution of the Cauchy problem for general linear partial differential equations

$$P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u(x,t) \equiv \sum_{0\leqslant k\leqslant m} P_k\left(\frac{\partial}{\partial x}\right)\frac{\partial^k u(x,t)}{\partial t^k} = 0, \tag{1}$$

$$-\infty < x < \infty, \quad 0 \leqslant t < \infty,$$

where $P(\lambda, w)$ is an arbitrary polynomial with constant coefficients of degree n in w and degree m in λ, $P_m(w) \not\equiv 0$, under the initial conditions

$$\partial^k u(x,0)/\partial t^k = 0, \qquad k = 0, 1, \dots, m-1, \tag{2}$$

has been the subject of study in the works of many authors (see ([1]), where further references are given). In all these works the question concerned the uniqueness of the solution of the Cauchy problem for (1) in the class of functions which, for all x, $-\infty < x < \infty$, satisfy certain estimates characterizing their growth as $|x| \to \infty$.

In the present paper we study the question of uniqueness of the solution of the Cauchy problem (1)–(2) in the class of functions satisfying a certain estimate only on one of the half-axes $x \geqslant 0$ or $x \leqslant 0$. No assumptions are made about the behavior of these functions on the other half-axis. It turns out that in this case as well one can give a complete description of the conditions (necessary and sufficient) for uniqueness of the solution of the Cauchy problem in one or another class of functions. Moreover, as in ([1]), we assume that solutions of the Cauchy problem for equation (1) have normal type with respect to t, i.e. that these solutions and all their derivatives entering into (1) grow in t, as $t \to \infty$, no faster than $\exp\{at\}$ with some $a > 0$.

The subsequent considerations concern the case when the estimates on functions in whose class the uniqueness questions are studied are prescribed on the half-axis $x \leqslant 0$; for $x \geqslant 0$ the study can be carried out by an analogous method.

For equation (1) let us construct the characteristic equation $P(\lambda, w) = 0$. The roots of this equation (not necessarily distinct) are of the form ([2]):

$$w_j(\lambda) = a_j^{(0)} \lambda^{q_j^{(0)}} + a_j^{(1)} \lambda^{q_j^{(1)}} + \dots,$$
$$j = 0, 1, \dots, n-1, \qquad a_j^{(0)} \neq 0, \qquad q_j^{(0)} > q_j^{(1)} > \dots \tag{3}$$

We assign to type T_1 those roots $w_j(\lambda)$ such that for any $A > 0$, $\operatorname{Re} w_j(\lambda) > A$ for sufficiently large σ.

Next we consider roots $w_j(\lambda)$ possessing the following property: for $\lambda = \sigma_0 + i\tau$ (σ_0 sufficiently large)

$$\operatorname{Re} w_j(\lambda) = B_j + o(1), \tag{4}$$

where $o(1) \to 0$ either as $\tau \to +\infty$ or as $\tau \to -\infty$.

Denote $B = \max B_j$, where B_j is defined in (4);

$$\operatorname{Re} w_j(\lambda) - B_j = \operatorname{Re} W_j(\lambda).$$

To type T' we assign the roots $w_j(\lambda)$ having property (4) with $B_j < B$.

To type T_2 we assign the roots $w_j(\lambda)$ having the following properties:

1) (4) holds with $B_j = B$;

2′) if $q_j^{(0)} > 0$, then

$$\operatorname{Re} W_j(\lambda) = c_j r^{-\beta_j}(1 + o(1)), \tag{5}$$

$c_j > 0$, $\beta_j > 0$, $\lambda = \tau_4 + i\tau$, $o(1) \to 0$ either as $\tau \to +\infty$ or as $\tau \to -\infty$;

2″) if $q_j^{(0)} \leq 0$, then $\operatorname{Re} W_j(\lambda) > 0$ for $\operatorname{Re}\lambda \geq \sigma_0$ and (5) holds.

Remark. In the case $q_j^{(0)} \leq 0$, the β_j in (5) for $\tau > 0$ and $\tau < 0$ may be different. In what follows we shall be interested in representation (5) for those τ for which β_j is largest.

To type T_3 we assign the roots $w_j(\lambda) \equiv \text{const}$ with $B_j = B$.

To type T_4 we assign the roots $w_j(\lambda)$ having the following properties:

1) (4) holds with $B_j = B$;

2) there exists a $-\pi/2 \leq \chi_j \leq \pi/2$ such that $\operatorname{Re} W_j(\lambda) < 0$ for $\lambda = \sigma_0 + \rho \exp\{i\chi_j\},\ \rho > \rho_0 > 0$.

To type T_5 we assign the roots $w_j(\lambda)$ having the following property:

$$\operatorname{Re} w_j(\lambda) = -c_j r^{\gamma_j}(1 + o(1)), \quad c_j > 0, \quad 0 < \gamma_j < 1, \quad \lambda = \sigma_0 + i\tau, \quad o(1) \to 0$$

either as $\tau \to +\infty$ or as $\tau \to -\infty$.

To type T_6 we assign the roots $w_j(\lambda)$ having the following property: for every $-\pi/2 < \chi_j < \pi/2$, for $\lambda = \sigma_0 + \rho \exp\{i\chi_j\}$,

$$\operatorname{Re} w_j(\lambda) = -c_j r(1 + o(1)), \quad c_j > 0, \quad o(1) \to 0$$

as $\rho \to \infty$.

To type T_7 we assign the roots $w_j(\lambda)$ having the following property: there exists a $-\pi/2 \leq \chi_j \leq \pi/2$ such that, for $\lambda = \sigma_0 + \rho \exp\{i\chi_j\}$,

$$\operatorname{Re} w_j(\lambda) = -c_j r^{\gamma_j}(1 + o(1)), \quad c_j > 0, \quad \gamma_j \geq 1, \quad o(1) \to 0$$

as $\rho \to \infty$.

Using the expansion of the roots $w_j(\lambda)$ in a Newton-Puiseux series (3), one can show that each root $w_j(\lambda)$ belongs to one and only one of the types $T_n,\ n = 1, \dots, 7,\ T'$.

Definition. We assign equation (1) to type $\Gamma_k,\ 1 \leq k \leq 7$, if there exists at least one root $w_j(\lambda)$ of type T_k, but none of the roots has type $T_l,\ 1 \leq l \leq k$.

Let us note that if the characteristic equation has a root of type T', then there is necessarily a root of one of the types T_2, T_3, T_4. Hence it follows that equations of types $\Gamma_1 - \Gamma_4$ may have roots of type T'.

Theorem 1. *Let equation (1) have type Γ_1, and let $h(x) > 0$ be a continuous function $(x \leq 0)$. Then, for uniqueness of the solution of the Cauchy problem (1)–(2) in the class of functions satisfying the estimates*

$$|D_x^k u(x,t)| \leq C \exp\{at - |x| h(x)\},$$

$$x \leq 0, \quad t \geq 0, \quad \alpha > 0, \quad k = 0, 1, \dots, n-1, \tag{6}$$

it is necessary and sufficient that $\sup h(x) = \infty$*.*

Theorem 2. *Let equation (1) have type* Γ_2*, and let* $h(t) > 0$ *be a continuous function decreasing for* $t \geq 0$*. Then, for uniqueness of the solution of the Cauchy problem (1)–(2) in the class of functions satisfying the estimates*

$$|D_x^k u(x,t)| \leq C \exp\{\alpha t\} \exp\left\{-B|x| - \int_0^{|x|} h(t)\, dt\right\}, \quad x \leq 0, \quad t \geq 0, \quad \alpha > 0, \tag{7}$$

$$k = 0, 1, \dots, n-1,$$

it is necessary and sufficient that

$$\int_0^{\infty} [h(t)]^{1+1/\beta}\, dt = \infty, \qquad \beta = \min_{\{j : w_j(\lambda) \in T_2\}} \beta_j.$$

Theorem 3. Let equation (1) have type Γ_3, and let $\beta(x) > 0$ be a monotone function ($x \leq 0$). Then, for uniqueness of the solution of the Cauchy problem (1)–(2) in the class of functions satisfying the estimates

$$\left|D_x^k u(x,t)\right| \leq \beta(x) \exp\{\alpha t - B|x|\},$$
$$x \leq 0, \quad t \geq 0, \quad \alpha > 0, \quad k = 0, 1, \dots, n-1,$$

it is necessary and sufficient that

$$\lim_{x \to -\infty} \beta(x) = 0.$$

Theorem 4. Let equation (1) have type Γ_4, and let $h(x) > 0$ be a continuous function ($x \leq 0$). Then, for uniqueness of the solution of the Cauchy problem (1)–(2) in the class of functions satisfying the estimates

$$\left|D_x^k u(x,t)\right| \leq C \exp\{\alpha t\} \exp\{-B|x| + |x| h(x)\},$$
$$x \leq 0, \quad t \geq 0, \quad \alpha > 0, \quad k = 0, 1, \dots, n-1,$$

it is necessary and sufficient that $\inf h(x) = 0$.

Theorem 5. Let equation (1) have type Γ_5, and let $h(t)$ be a continuous function increasing for $t \geq 0$. Then, for uniqueness of the solution of the Cauchy problem (1)–(2) in the class of functions satisfying the estimates

$$\left|D_x^k u(x,t)\right| \le C \exp\left\{\alpha t + \int_0^{|x|} h(t)\,dt\right\}, \quad x \le 0, \quad t \ge 0, \quad \alpha > 0, \quad k = 0, 1, \dots, n-1, \tag{8}$$

it is necessary and sufficient that

$$\int^{\infty} [h(t)]^{1-1/\gamma}\,dt = \infty, \quad \gamma = \min_{\{j: w_j(\lambda) \in T_5\}} \gamma_j.$$

Theorem 6. Let equation (1) have type Γ_6, and let $h(t) > 0$ be a continuous function increasing for $t \ge 0$. Then, in the class of functions satisfying the estimates

$$\left|D_x^k u(x,t)\right| \le C \exp\left\{\alpha t + \int_0^{|x|} h(t)\,dt\right\}, \quad x \le 0, \quad t \ge 0, \quad \alpha > 0, \quad k = 0, 1, \dots, n-1,$$

where

$$\int^{\infty} [h(t)]^{-\varepsilon}\,dt = \infty$$

for some $\varepsilon > 0$, the solution of the Cauchy problem (1)–(2) is identically zero.

Theorem 7. Let equation (1) have type Γ_7. Then the Cauchy problem (1)–(2) can have only the trivial solution.

The proofs of the stated theorems are carried out according to the scheme of the proofs in ([1]): with the aid of the Laplace transform the problem is reduced to investigating the question of the existence of a nontrivial solution, analytic in the half-plane $\operatorname{Re}\lambda \ge \alpha$ and satisfying there certain estimates, of the corresponding ordinary differential equation whose coefficients are polynomials in λ; to solve the latter question, certain theorems of function theory are used.

Example. Consider the Cauchy problem for the equation

$$\partial u/\partial t = a\,\partial^2 u/\partial x^2, \tag{9}$$

$$u(x, 0) = 0. \tag{10}$$

The characteristic equation here has the form

$$\lambda = a w^2.$$

Denote

$$\varphi = \arg a, \quad -\pi < \varphi \leq \pi; \qquad \theta = \arg \lambda, \quad -\pi < \theta < \pi/2.$$

$$\operatorname{Re} w_0(\lambda) = \sqrt{\frac{r}{|a|}} \cos \frac{\theta - \varphi}{2},$$

$$\operatorname{Re} w_1(\lambda) = \sqrt{\frac{r}{|a|}} \cos \left(\frac{\theta - \varphi}{2} + \pi \right).$$

1. $-\pi/2 < \varphi < \pi/2$. Here $w_0(\lambda) \in T_1$. Then, for the uniqueness of the solution of the Cauchy problem (9)–(10) in the class of functions (6), it is necessary and sufficient that $\sup h(x) = \infty$.
2. $\varphi = \pm\pi/2$. In this case $w_0(\lambda) \in T_2$, $\beta = 1/2$; $w_1(\lambda) \in T_4$.

For the uniqueness of the solution of the Cauchy problem (9)–(10) in the class of functions (7), it is necessary and sufficient that

$$\int^{\infty} [h(t)]^3 \, dt = \infty.$$

3. $-\pi < \varphi < -\pi/2, \ \pi/2 < \varphi \leq \pi$. In this case $w_0(\lambda), w_1(\lambda) \in T_5$, $\gamma = 1/2$.

For the uniqueness of the solution of the Cauchy problem (9)–(10) in the class of functions (8), it is necessary and sufficient that

$$\int^{\infty} [h(t)]^{-1} dt = \infty.$$

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Note: Figure translations are in progress. See original paper for figures.