ON THE DARBOUX PROBLEM FOR HYPERBOLIC EQUATIONS

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 517.946

MATHEMATICS

A. M. NAKHUSHEV

ON THE DARBOUX PROBLEM FOR HYPERBOLIC EQUATIONS

(Presented by Academician S. L. Sobolev on 11 V 1970)

Consider the equation

$$Lu \equiv u_{yy} - k(y)u_{xx} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y), \qquad (1)$$

where $k(y) > 0$ for $y \neq 0$ and may vanish at $y = 0$.

Let D be a simply connected domain in the plane of the independent variables x, y, bounded by the characteristics AC and CB of equation (1), issuing from the point $C({}^1/_2, y_c)$, $y_c < 0$, and by the segment $AB: \ 0 < x < 1$ of the axis $y = 0$.

With respect to the coefficients k and b of equation (1), we shall assume that they are continuous in the closure $\overline{D} = D \cup \partial D$ of the domain D, and with respect to a and c, that they are continuous in $\overline{D}$ together with the derivative with respect to x.

Denote by W the set of functions $u = u(x,y)$ of the class $C(\overline{D}) \cap C^2(D) \cap W_2^1(D) \cap W_2^1(\partial D)$, where $W_2^1(D)$ is the Sobolev space, for which $Lu \in L_2(D)$ and the boundary conditions are satisfied

$$u\big|_{AB\cup BC} = 0, \quad \text{or} \quad u_y\big|_{AB} = 0, \quad u\big|_{BC} = 0. \qquad (2)$$

Darboux problem. Find a function $u \in W$ satisfying equation (1) in the domain D.

The uniqueness and existence of the nonhomogeneous Darboux problem: $u_y|_{AB} = \nu(x)$, $u|_{BC} = \psi(y)$, for equation (1) in the case when $k(y) = -y^m$, $m \equiv 1(\mathrm{mod} 2)$, a and $b \in C^3(\overline{D})$, $c, f \in C^1(\overline{D})$, and, in addition, for $m \geq 2$ the function a is representable in the form $a = |y|^n a_1(x,y)$ with $a_1 \in C(\overline{D})$ and $n > m/2 - 1$, were established by Gellerstedt [1] by the method of the Green-Hadamard function.

Theorem. Suppose the coefficients of equation (1) satisfy one of the following conditions:

1. $k(0) \neq 0$ or $a(x,0) > 0$ for $0 \leq x \leq 1$;
2. $a/k,\ b^2/k \in C(\overline{D})$, $c(x,0) < 0$ for $0 \leq x \leq 1$;
3. $a/k,\ b^2/k,\ a_x/k,\ c/k,\ c_x/k \in C(\overline{D})$;
4. $k/a,\ b^2/a \in C(\overline{D})$, $a > 0$, $c(x,0) < 0$ for $y \neq 0,\ 0 \leq x \leq 1$;
5. $k/a,\ b^2/a,\ c/k,\ c_x/k \in C(\overline{D})$, $a > 0$ for $y \neq 0,\ a_x \geq 0$.

Then an a priori estimate of the form holds

$$\|u\|_{++} \leq C\|Lu\|_{+}, \qquad \forall u \in W,$$

where $\|\cdot\|_{++},\ \|\cdot\|_{+}$ are certain positive norms, and C is a constant independent of u.

Proof. Introduce the operator

$$L_\mu v \equiv v_{yy} - kv_{xx} + a_\mu v_x + bv_y + c_\mu v,$$

where

$$\mu \equiv \text{const} < 0, \qquad a_\mu = a - 2\mu k, \qquad c_\mu = c + a\mu - k\mu^2.$$

It is easy to verify that if $u = v\exp(\mu x)$ and $u \in W$, then $Lu = \exp(\mu x)L_\mu v$ and $v \in W$. For any function $d = d(x) \in C^1(\overline{D})$ and $v \in W$ the identity is valid –

$$\begin{aligned}2(dv_x, L_\mu v)_0 = 2\int_D dv_x L_\mu v\, dD = \int_D \left[d_x(kv_x^2 + v_y^2) + 2da_\mu v_x^2 + 2dbv_{xv}y\right] dD \\ - \int_D (dc_\mu)^2_{xv} dD + \int_{\partial D} dc_\mu x_n v^2\, dS \\ + \int_{\partial D} d\left[-(kv_x^2 + v_y^2)x_n + 2y_{nvxv}y\right] dS = \sum_{j=1}^{4} I_j,\end{aligned} \tag{3}$$

where x_n and y_n are the direction cosines of the outward normal $n = (x_n, y_n)$ to the boundary of the domain D.

Since the roots of the characteristic equation $\lambda(\lambda + x_n + kx_n) = 0$, corresponding to the quadratic form $-kx_n\xi^2 + 2y_n\xi\eta - x_n\eta^2$ on the characteristic AC, are nonnegative, and on the characteristic BC, in view of (2), $v_x = v_{nx}n,\ v_y = v_{ny}n$, it is not difficult to verify that $I_4 \geq 0$ for all $v \in W$, if $d(x) \geq 0$ in $\bar{D}$.

Suppose that $d = \exp(\alpha x)$, where $\alpha = \text{const} > 0$. From (3) and the elementary inequality $2bv_{xv}y \geq -\varepsilon b^2 v_x^2 - \frac{1}{\varepsilon} v_y^2$, valid for any $\varepsilon > 0$, we have

$$\int_D d\left[(\alpha k + 2a_\mu - \varepsilon b^2)v_x^2 + (\alpha - 1/\varepsilon)v_y^2\right] dD + I_2 + I_3 \leq \varepsilon \|v_x \sqrt{d\omega}\|_0^2 + C\|\omega^{-1/2} L_\mu v\|_0^2, \tag{4}$$

where $\omega = \omega(x, y)$ is any nonnegative function of the class $C(\bar{D})$, and $\|\cdot\|_0$ is the norm in the space $L_2(D)$. Here and below C denotes a certain positive constant that does not depend on v.

For convenience we introduce positive norms by the formulas

$$\|v\|_{1,\varphi,\psi} = \left[\int_D (\varphi v_x^2 + v_y^2 + \psi v^2)\, dD\right]^{1/2}, \qquad \|v\|_{1,\varphi,1} \equiv \|v\|_{1,\varphi},$$

$$\|v\|_{0,\varphi} = \left(\int_D \frac{v^2}{\varphi}\, dD\right)^{1/2}, \qquad \|v\|_{j,1} \equiv \|v\|_j, \quad j = 0, 1.$$

Let us rewrite inequality (4) in the form

$$\int_D d\left[a_\mu^\alpha v_x^2 + (\alpha - 1/\varepsilon)v_y^2 - c_\mu^\alpha v^2\right] dD \leq C\|L_\mu v\|_{0,\omega} - I_3, \tag{5}$$

where

$$a_\mu^\alpha = 2a + (\alpha - 4\mu)k - \varepsilon(b^2 + \omega),$$

$$c_\mu^\alpha = \alpha c + c_x + (\alpha a + a_x)\mu - \alpha k \mu^2 = \alpha c_\mu + c_{\mu x}.$$

Let condition 1 of the theorem be satisfied, for example, $a(x, 0) > 0$. Put

$$\mu < -\min_{0 \leq x \leq 1} c(x, 0)/a(x, 0).$$

Then, for sufficiently small values of ε, there is a number $\delta < 0$ such that

$$c_\mu < 0, \qquad a_\mu^\alpha > 0, \qquad \forall (x, y) \in \bar{D} \cap (\delta \leq y \leq 0).$$

Therefore there exists a number μ_0, independent of α, such that

$$c_{\mu_0} < 0, \qquad a_{\mu_0}^\alpha > 0, \qquad \forall \alpha > 0, \quad (x, y) \in \bar{D} \tag{6}$$

and a number α_0, depending on μ_0, such that

$$c_{\mu_0}^{\alpha_0} < 0, \qquad \forall (x,y) \in \bar{D}. \tag{7}$$

Taking into account (6) and (7), from (5), with $\omega \equiv 1$, we obtain the energy inequality

$$\|v\|_1 \leq C\|L_{\mu_0} v\|_0, \qquad \forall v \in W. \tag{8}$$

Similarly, but even more simply, estimate (8) is proved when $k(0) \neq 0$, i.e., equation (1) is strictly hyperbolic.

If condition 2 holds, then for $\omega \equiv k$, obviously, there exists a number μ_0, independent of α, such that

$$c_{\mu_0} < 0, \qquad a_{\mu_0}^{\alpha}/k = \alpha - 4\mu_0 + (2a - \varepsilon b^2 - \varepsilon k)/k > 0, \qquad \forall (x,y) \in \overline{D}$$

and a number α_0 such that (7) is true. Consequently, by virtue of (5),

$$\|v\|_{1,k} \leq C\|L_{\mu_0} v\|_{0,k}, \qquad \forall v \in W.$$

Suppose now that condition 3 is fulfilled. Then for $\omega \equiv k$ there is a number μ_0, independent of α, such that $c_{\mu_0}/k < 0$, $a_{\mu_0}^{\alpha}/k > 0$ in $\overline{D}$, and a number α_0 such that $c_{\mu_0}^{\alpha_0}/k < 0$ in $\overline{D}$. Therefore, in accordance with (5), we have

$$\|v\|_{1,k,k} \leq C\|L_{\mu_0} v\|_{0,k}, \qquad \forall v \in W.$$

In exactly the same way we verify that

$$\forall v \in W, \qquad \|v\|_{1,\alpha} \leq C\|L_{\mu_0} v\|_{0,\alpha}$$

and

$$\|v\|_{1,\alpha,k} \leq C\|L_{\mu_0} v\|_{0,\alpha}$$

under conditions 4 and 5, respectively; this completes the proof of the theorem.

From the theorem just proved follows the uniqueness of the regular solution of the Darboux problem and the existence of a weak solution of the adjoint problem

$$v_{yy} - kv_{xx} - (av)_x - (bv)_y + cv = f, \qquad v|_{AB \cup AC} = 0$$

in the functional spaces corresponding to the a priori estimate.

It is not difficult to show that the basic inequality (5) guarantees uniqueness of the solution of the Darboux problem under more general assumptions concerning the coefficients of equation (1). However, it should be noted that *some of the conditions of the theorem are essential, and their violation may lead even to nonuniqueness of the solution of the Darboux problem.*

Example 1. The function

$$u = \sin\left(x - 1 + \int_y^0 \exp\frac{1}{2t}\,dt\right)$$

in the domain D is a solution of the homogeneous Darboux problem:

$$u|_{BC} = 0, \qquad u_y|_{AB} = 0$$

for the equation

$$u_{yy} - \exp\left(\frac{1}{y}\right)u_{xx} - \frac{1}{2y^2}\exp\left(\frac{1}{2y}\right)u_x = 0.$$

Example 2. The general solution of the equation

$$u_{yy} - y^2 u_{xx} - u_x = 0 \tag{9}$$

in the class of functions

$$u \in C^1(D \cup AB) \cap C^2(D),$$

possessing the property that

$$u(x,0) \quad \text{and} \quad u_y(x,0) \in C^1(0 \le x \le 1) \cap C^2(0 < x < 1),$$

is given by formula (2)

$$u(x,y) = \tau\left(x + \frac{1}{2}y^2\right) + \frac{1}{2}y\int_0^1 \nu\left(x + \frac{1}{2}y^2 - y^2 t\right)\frac{dt}{\sqrt{t}}, \tag{10}$$

where $\tau(x)$ and $\nu(x)$ are arbitrary functions from

$$C^1(0 \le x \le 1) \cap C^2(0 < x < 1).$$

Using formula (10), it is not difficult to verify the validity of the following assertion: *the homogeneous problem corresponding to the nonhomogeneous Darboux problem*

$$u_y|_{AB} = \nu(x), \qquad u|_{BC} = \psi(x)$$

for equation (9) has an infinite number of linearly independent solutions, while the nonhomogeneous problem is solvable if and only if

$$\nu(x) = \psi_*(x) \equiv \sqrt{1-x}\,\psi'\left(\frac{1}{2} + \frac{x}{2}\right).$$

For equation (9) the Darboux problem is correctly posed when u is prescribed not on BC, but on AC, which indicates the nonequivalence of these characteristics, or when on AB not u_y, but u is prescribed; and, finally, the following boundary-value problem

$$u|_{BC} = \psi(x), \qquad \lim_{y\to 0} \frac{u_y - \psi_*(x)}{y} = \nu(x), \qquad 0 < x < 1,$$

provided, for example, that

$$\psi(x) \in C^4\left(\frac{1}{2} \le x \le 1\right), \qquad \nu(x) \in C^1(0 \le x \le 1) \cap C^2(0 < x < 1).$$

It is curious to note that equation (9) was given by A. V. Bitsadze ([2]) as an example of an equation for which the Cauchy problem with initial data on the line of parabolic degeneration is well posed. Consequently, unlike strictly hyperbolic equations, or degenerating equations with noncharacteristic power-law degeneration of order less than two, the well-posedness of the Cauchy problem does not imply, in general, the well-posedness of the corresponding Darboux problems if the order of degeneration is greater than or equal to two.

In conclusion, we note that, in connection with the examples presented here, the question arises whether nonuniqueness of the solution of the Darboux problem can affect the uniqueness of the well-known Tricomi problem. For example, it is easy to verify that for the equation

$$\operatorname{sign} y \cdot y^2 u_{xx} + u_{yy} - u_x = 0$$

this does not occur.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR
Novosibirsk

Received
30 IV 1970

References

[1] S. Gellerstedt, *Ark. Mat., Astron., Fys.*, **25A**, No. 29 (1937).
[2] A. V. Bitsadze, *Equations of Mixed Type*, Moscow, 1959.

Note: Figure translations are in progress. See original paper for figures.