THE HILBERT PARALLELEPIPED CANNOT BE DECOMPOSED INTO A COUNTABLE UNION OF CLOSED SUBSETS DISTINCT FROM IT, WHOSE PAIRWISE INTERSECTIONS ARE WEAKLY INFINITE-DIMENSIONAL

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Abstract

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THE HILBERT PARALLELEPIPED CANNOT BE DECOMPOSED INTO A COUNTABLE UNION OF CLOSED SUBSETS DISTINCT FROM IT, WHOSE PAIRWISE INTERSECTIONS ARE WEAKLY INFINITE-DIMENSIONAL

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The aim of the present note is to prove the assertion announced in the title.

We shall call a space X **weakly infinite-dimensional** if, for every countable system of pairs of closed sets A_{+n} and A_{-n} such that $A_{+n} \cap A_{-n} = \emptyset$, one can choose partitions* C_n between A_{+n} and A_{-n} whose intersection is empty:

$$\bigcap_{n=1}^{\infty} C_n = \emptyset \tag{1}$$

Obviously, a closed subset of a weakly infinite-dimensional space will also be weakly infinite-dimensional.

The following two lemmas require the concept of normal adjoining, introduced for another purpose by Yu. M. Smirnov (see ([2])):

A set N of a topological space X **normally adjoins** its complement $M = X \setminus N$ if any two disjoint closed subsets of M have disjoint neighborhoods open in X.

Lemma 1. *Every subset of type G_δ of a normal space normally adjoins its complement.*

We omit the simple proof of this lemma.

Lemma 2. *If, in a normal space X, the subset $X \setminus M$ normally adjoins the weakly infinite-dimensional countably paracompact subset M, then for every countable system of pairs of closed sets (A_{+n}, A_{-n}) such that $A_{+n} \cap A_{-n} = \emptyset$, one can choose partitions C_n between A_{+n} and A_{-n}, the intersection of which has no common points with M:*

$$M: \quad M \cap \bigcap_{n=1}^{\infty} C_n = \emptyset.$$

For any pair (A_{+n}, A_{-n}) take neighborhoods U_{+n} and U_{-n}, open in X, such that $\overline{U}_{+n} \cap \overline{U}_{-n} = \emptyset$. For the countable system of pairs of closed subsets in M, $(M \cap \overline{U}_{+n}, M \cap \overline{U}_{-n})$, one can choose partitions C'_n in M between $M \cap \overline{U}_{+n}$ and $M \cap \overline{U}_{-n}$, whose intersection is empty:

$$\bigcap_{n=1}^{\infty} C'_n = \emptyset.$$

In the countably paracompact space M, the partitions C'_n can be enlarged to such open subsets L_n of M that

$$\bigcap_{n=1}^{\infty} L_n = \emptyset,$$

and L_n does not intersect $M \cap \overline{U}_{+n}$ and $M \cap \overline{U}_{-n}$ for any n. The sets C'_n are partitions between $M \cap \overline{U}_{+n}$ and $M \cap \overline{U}_{-n}$, and, consequently, $M \setminus C'_n = M_{+n} \cup M_{-n}$, where M_{+n} and M_{-n} are disjoint neighborhoods, open in M, of the sets $M \cap \overline{U}_{+n}$ and $M \cap \overline{U}_{-n}$. Since the sets closed in M, $M_{+n} \setminus L_n$ and $M_{-n} \setminus L_n$, do not pe–

* A closed subset C of a space X is called a partition between the sets A and B if the complement $X \setminus C$ is decomposed into such disjoint open sets G and H that $A \subset G$, $B \subset H$.

intersect, they have disjoint open neighborhoods V_{+n} and V_{-n} in X. The sets

$$C_n = X \setminus \left(U_{+n} \cup (V_{+n} \setminus \overline{U}_{-n}) \cup U_{-n} \cup (V_{-n} \setminus \overline{U}_{+n})\right)$$

are partitions between A_{+n} and A_{-n} in the space X, and

$$M \cap \bigcap_{n=1}^{\infty} C_n = \emptyset.$$

Lemma 3. *If a normal space X is the union of its weakly infinite-dimensional countably paracompact subsets X_i, where $X \setminus X_i$ is normally situated with respect to X_i, $i = 1, 2, \ldots$, then the space X is weakly infinite-dimensional.*

Let, in the infinite matrix

$$\begin{gathered} n_{11}, n_{12}, n_{13}, \dots \\ n_{21}, n_{22}, n_{23}, \dots \\ n_{31}, n_{32}, n_{33}, \dots \\ \dots\dots\dots \end{gathered}$$

each natural number occur exactly once. If (A_{+n}, A_{-n}), $n = 1, 2, \dots$, is a countable system of pairs of closed disjoint sets in X, then we must construct partitions C_n in X between A_{+n} and A_{-n} such that

$$\bigcap_{n=1}^{\infty} C_n = \emptyset.$$

By Lemma 2, for the countable system of pairs $(A_{+n_{ki}}, A_{-n_{ki}})$, $i = 1, 2, \dots$, one can choose partitions $C_{n_{ki}}$ in X between $A_{+n_{ki}}$ and $A_{-n_{ki}}$ such that

$$X_k \cap \bigcap_{i=1}^{\infty} C_{n_{ki}} = \emptyset.$$

Then $\{C_{n_{ki}}\}_{k,i=1}^{\infty}$ will be the required system of partitions.

In the Hilbert parallelepiped

$$I_{\text{alef}}^{\aleph_0} = [-1, 1]_{\text{alef}}^{\aleph_0}$$

any subset

$$P = \prod_{n=1}^{\infty} [\alpha_n, \beta_n]$$

will also be called a parallelepiped, and of its subsets

$$P_{-i} = \alpha_i \times \prod_{n \neq i} [\alpha_n, \beta_n] \quad \text{and} \quad P_{+i} = \beta_i \times \prod_{n \neq i} [\alpha_n, \beta_n]$$

we shall say that they are its opposite faces.

Let

$$P \subset \bigcup_{i=1}^{\infty} \Phi_i,$$

where Φ_i are closed subsets of the space $I^{\aleph_0}$, $P \not\subset \Phi_i$ for no i, and the intersections $\Phi_i \cap \Phi_j$ are weakly infinite-dimensional for any distinct i and j. Note that the set $X \setminus M$ is normally situated with respect to

$$M = \bigcup_{i \neq j} (\Phi_i \cap \Phi_j)$$

(see Lemma 1), and the set M is weakly infinite-dimensional (see Lemma 3).

Lemma 4. *For any i, at least one of the summands Φ_k contains a continuum joining the opposite faces P_{-i} and P_{+i} of the parallelepiped P.*

By symmetry it suffices to restrict ourselves to the case $i = 1$. By Lemma 2 there exist partitions in P between the sets P_{-n} and P_{+n}, $n = 2, 3, ...$, such that the set

$$C = \bigcap_{n=2}^{\infty} C_n$$

does not meet the set M. To prove the lemma, it is enough to find at least one component of connectedness K of the compactum C joining P_{-1} with P_{+1}. Indeed, since $K \cap M = \emptyset$, the continuum K is decomposed into the union of pairwise disjoint closed sets $K \cap \Phi_j$. Hence, by a theorem of Sierpiński ([3]),

$$K = K \cap \Phi_k$$

for some natural k, i.e. $K \subset \Phi_k$, which is what the lemma requires.

We shall show that such a component can be found. Indeed, if there were no such components, then it is not difficult to show that the compactum C can be split into the sum of disjoint open-and-closed-in-C sets C_+ and

C_- in such a way that $C_+ \cap P_{-1} = \emptyset$ and $C_- \cap P_{+1} = \emptyset$. It is clear that the closed sets $P_{+1} \cup C_+$ and $P_{-1} \cup C_-$ do not intersect. Hence they can be separated in the parallelepiped P by a closed partition C_1. Thus the opposite faces P_{-n} and P_{+n}, $n = 1, 2, ...$, of the parallelepiped P have turned out to be separable by partitions C_n with empty intersection

$$\bigcap_{n=1}^{\infty} C_n = \emptyset,$$

which is impossible (see (1), p. 75). Consequently, contrary to the assumption, there exists a component of connectedness K of the compactum C joining the face P_{-1} with the face P_{+1}, as was to be proved.

Lemma 5. *No summand Φ_k contains any parallelepiped N two of whose faces are parallel to the faces P_{-i} and P_{+i}, while the remaining ones lie on the faces of the parallelepiped P, i.e.*

$$\Phi_k \supset N[a_i, b_i] \times \prod_{n \neq i} (\alpha_n, \beta_n),$$

where $[a_i, b_i] \subset [\alpha_i, \beta_i]$.

Of course, we may again assume that $i = 1$. Suppose that, for example, the summand Φ_1 contains entirely some parallelepiped N of the indicated form. Then the union of the closed differences $\Phi_k \setminus N^{\circ *}$, $k \neq 1$, together with the first summand Φ_1, covers the parallelepiped P; none of them contains P, and all their pairwise intersections are weakly infinite-dimensional. Therefore we may assume that the given decomposition is such that some parallelepiped of the indicated form does not intersect any of the summands Φ_k, $k \neq 1$. Since the closed set Φ_1 does not contain P, there will be found in P a parallelepiped

$$Q = \prod_{k=1}^{\infty} [\gamma_k, \delta_k],$$

where $[\gamma_k, \delta_k] = [\alpha_k, \beta_k]$ for $k \neq k_i$, $i = 1, 2, \dots, m$, and $Q \cap \Phi_1 = \emptyset$. The parallelepiped

$$R = [\alpha_1, \beta_1] \times \prod_{n=2}^{\infty} [\gamma_n, \delta_n],$$

obviously contains Q. R does not lie in any of the summands Φ_k, $k \neq 1$, since otherwise N would intersect one of them. Nor does R lie in Φ_1, since Q does not lie in Φ_1. Hence, by Lemma 4, in R there exists a continuum K lying in some Φ_k and joining the faces of the parallelepiped R that lie on the faces P_{-1} and P_{+1}. This continuum K necessarily intersects N and, consequently, $\Phi_k \cap N \neq \emptyset$, which entails $k = 1$, since $K \subset \Phi_1$. But the continuum K also intersects Q, so that $Q \cap \Phi_1 \neq \emptyset$, contradicting the choice of the parallelepiped Q.

We proceed to the proof of the proposition formulated in the title.

Let

$$I^{\aleph_0} = \bigcup_{k=1}^{\infty} \Phi_k,$$

where the closed subsets Φ_k do not coincide with $I^{\aleph_0}$ and the intersections $\Phi_i \cap \Phi_j$ are weakly infinite-dimensional for $i \neq j$. Since $I^{\aleph_0}$ is a complete metric space, all the summands Φ_k cannot be nowhere dense. Hence at least one of the summands, let it be Φ_1, contains entirely some parallelepiped

$$Q = \prod_{k=1}^{\infty} [a_k, b_k],$$

where for $k > n$ we have $[a_k, b_k] = [-1, 1]$, and $-1 \leq a_k < b_k \leq 1$ for $k = 1, 2, \ldots, n$. Put

$$P^{(i)} = \prod_{k=1}^{\infty} [a_k^{(i)}, b_k^{(i)}], \quad i = 1, 2, \ldots, n+1,$$

where $[a_k^{(i)}, b_k^{(i)}] = [-1, 1]$ for $k \geq i$, and $[a_k^{(i)}, b_k^{(i)}] = [a_k, b_k]$ for $k < i$. It is clear that $P^{(1)} = I^{\aleph_0}$, $P^{(n+1)} = Q$ and

$$P^{(i+1)} = [a_i, b_i] \times \prod_{k \neq i} [a_k^{(i)}, b_k^{(i)}].$$

By Lemma 5 the parallelepiped $P^{(2)}$ is not contained in any of the summands Φ_k. Again by Lemma 5 we conclude that $P^{(3)}$ is not contained entirely in any of the summands Φ_k. Continuing these

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$$* \, N = (a_i, b_i) \times \prod_{n \neq i} [\alpha_n, \beta_n].$$

reasoning, at the n-th step we obtain that Q is not contained in any of the summands Φ_k, which contradicts the choice of the parallelepiped Q. The proposition is proved.

With the aid of the assertion just proved it is not difficult to prove the following theorem.

Theorem. *If M is a connected topological space, every point x of which has a neighborhood homeomorphic to the product $R^{\aleph_0}$, where R is the real line, then M cannot be decomposed into a countable union of closed subsets, different from all of M, whose pairwise intersections are weakly infinite-dimensional.*

Let

$$M = \bigcup_{k=1}^{\infty} \Phi_k,$$

where the sets Φ_k are closed, $\Phi_k \neq M$ for no k, and the intersection $\Phi_i \cap \Phi_j$ is weakly infinite-dimensional whenever $i \neq j$. Let $x \in M$, and let Ox be a neighborhood of the point x homeomorphic to the space $R^{\aleph_0}$; for simplicity we shall assume that $Ox = R^{\aleph_0}$. We shall prove that Ox necessarily lies in one of the summands Φ_k.

Indeed,

$$Ox \subset \bigcup_{n=1}^{\infty} [-n, n]^{\aleph_0},$$

and since $[-n, n]^{\aleph_0}$, by what has already been proved, has no decomposition of the indicated kind, there exists a summand Φ_{k_n} not containing the set $[-n, n]^{\aleph_0}$. We may assert that for every n we have $k_n = k_1$; otherwise, from the inequality

$$[-1, 1]^{\aleph_0} \subset \Phi_{k_1} \cap \Phi_{k_n}$$

it would follow that the Hilbert cube $[-1, 1]^{\aleph_0}$ is weakly infinite-dimensional, which is a contradiction (see (1), p. 75). Hence $k_n = k_1$ for every n, and consequently $Ox \subset \Phi_{k_1}$. Thus $U = \operatorname{Int} \Phi_{k_1} \neq \emptyset$. Since the space M is connected, $\operatorname{Fr} U \neq \emptyset$. Now let $y \in \operatorname{Fr} U$, and let Oy be a neighborhood of the point y homeomorphic to the space $R^{\aleph_0}$; for simplicity we shall assume that $Oy = R^{\aleph_0}$. We already know that there exists a summand Φ_{k_2} containing Oy. It is clear that $k_2 \neq k_1$, since otherwise we would have $Oy \subset \Phi_{k_1}$, i.e. $y \in U$, which contradicts the choice of the point y. There exists an index n such that

$$[-n, n]^{\aleph_0} \cap U \neq \emptyset \quad \text{and} \quad [-n, n]^{\aleph_0} \not\subset \Phi_{k_1}.$$

Indeed, if $[-n, n]^{\aleph_0} \subset M \setminus U$ for every n, then

$$Oy \subset \bigcup_{n=1}^{\infty} [-n, n]^{\aleph_0} \subset M \setminus U,$$

and this entails $Oy \cap U = \emptyset$, which is a contradiction. On the other hand, if $[-n, n]^{\aleph_0} \subset \Phi_{k_1}$ for every n, then

$$Oy \subset \bigcup_{n=1}^{\infty} [-n, n]^{\aleph_0} \subset \Phi_{k_1},$$

which is a contradiction. Put

$$F_1 = [-n, n]^{\aleph_0} \cap \Phi_{k_1}, \qquad F_2 = [-n, n]^{\aleph_0} \setminus U.$$

It is clear that

$$F_1 \cup F_2 = [-n, n]^{\aleph_0}, \qquad F_1 \neq [-n, n]^{\aleph_0}, \qquad F_2 \neq [-n, n]^{\aleph_0}.$$

Moreover, the intersection $F_1 \cap F_2$ is weakly infinite-dimensional, since it is a closed subset of the weakly infinite-dimensional set $\Phi_{k_1} \cap \Phi_{k_2}$. This contradicts the assertion formulated in the title. Consequently, a decomposition of the space of the indicated kind is impossible.

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Note: Figure translations are in progress. See original paper for figures.