ON A DIFFERENTIAL GAME OF APPROACH

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Abstract

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The present work continues the investigations ($^{1-15}$). It generalizes the results from (^{12}) to the case when the control actions u and v in the right-hand side of the equations of motion are not separated. The basis of these results, as in (^{12}), is formed by the concept of an extremal strategy and the concept of absorption of the target set, modified for application to the class of problems under discussion.

Consider the following differential game of approach with a prescribed set $\mathcal{M}$. Let the controlled system be described by the vector differential equation

$$\dot{x} = f(t, x, u, v). \tag{1}$$

Here x is the n-dimensional phase vector of the system; t is time; u and v are r-dimensional vectors of control actions, subject to the first and second players respectively and constrained by the condition

$$u \in \mathcal{U}, \qquad v \in \mathcal{V}, \tag{2}$$

where $\mathcal{U}$ and $\mathcal{V}$ are bounded and closed sets. The vector-function f in (1) is continuous in all arguments and satisfies a Lipschitz condition with respect to x.

In the space $\{x\}$ a closed set $\mathcal{M}$ is given. The objective of the first player is to bring the point $x[t]$ onto $\mathcal{M}$ no later than some fixed time ϑ; the task of the second player is to prevent this. The initial position $\{t_0, x_0\}$ of the game is given.

Let us formulate the problem in the class of **mixed approximation strategies** U_a and V_a. A strategy U_a is specified by a system of sets $\{\mu(du)\}_{[t,x,\Gamma]}$, consisting of measures μ normalized on $\mathcal{U}$. Such a set $\{\mu(du)\}_{[t,x,\Gamma]}$ for each possible Δ-partition $\Gamma\{\tau_i \leq t < \tau_{i+1},\ i = 0, \dots, m;\ \tau_0 = t_0,\ \tau_m = \vartheta,\ \max(\tau_{i+1} - \tau_i) = \Delta > 0\}$ is put in correspondence with each possible position $\{t, x\}$. A strategy V_a is specified by a system of sets $\{\nu(dv)\}_{[t,x,\Gamma]}$, consisting of measures ν normalized on $\mathcal{V}$; such a set $\{\nu(dv)\}_{[t,x,\Gamma]}$ for each possible Δ-partition Γ is put in correspondence with each possible position $\{t, x\}$. The correspondence between the strategies and the sets defining them will be denoted symbolically as follows: $U_a \div \{\mu(du)\}_{[t,x,\Gamma]}$ and $V_a \div \{\nu(dv)\}_{[t,x,\Gamma]}$. Suppose a pair of strategies $\{U_a, V_a\}$ and a pair of Δ-partitions $\Gamma_u\{\tau_i^{(u)}\}$ and $\Gamma_v\{\tau_j^{(v)}\}$ have been chosen. By a **motion** $x[t] = x[t, t_0, x_0; U_a, \Gamma_u; V_a, \Gamma_v]$ generated by these strategies under the chosen Δ-partitions, we shall mean any continuous function $x[t]$ satisfying the initial condition $x[t_0] = x_0$ and the equalities

$$\dot{x}[t] = \iint f(t, x[t], u, v)\, \mu(du)_{[\tau_i, x[\tau_i], \Gamma_u]}\, \nu(dv)_{[\tau_j, x[\tau_j], \Gamma_v]} \tag{3}$$

$$\left(\tau_i^{(u)} < t < \tau_{i+1}^{(u)}, \quad \tau_j^{(v)} < t < \tau_{j+1}^{(v)}\right),$$

where the symbols $\mu(du)_{[\tau,x,\Gamma]}$ and $\nu(dv)_{[\tau,x,\Gamma]}$ denote some measures μ and ν from the sets $\{\mu\}_{[\tau,x,\Gamma]}$ and $\{\nu\}_{[\tau,x,\Gamma]}$, respectively.

Let us consider two more special strategies, U_τ and V_τ, which we shall call trivial. The strategy U_τ, paired with some strategy V_a, for a chosen Δ-partition Γ, determines the motion $x[t] = x[t, t_0, x_0; U_\tau; V_a, \Gamma]$ as an absolutely continuous function $x[t]$ which satisfies the initial condition $x[t_0] = x_0$ and, for almost all $t \in [t_0, \vartheta]$, satisfies the contingency

$$\dot{x}[t] \in \mathcal{F}_u(t, x[t]; \nu_{[\tau_i, x[\tau_i], \Gamma]}) \qquad (\tau_i < t < \tau_{i+1}). \tag{4}$$

Here the symbol $\mathcal{F}_u(t, x, \nu)$ denotes the convex hull of the set of vectors

$$f_u = \int f(t, x, u, v) \nu(dv), \qquad u \in \mathcal{U}. \tag{5}$$

The strategy V_τ, paired with the strategy U_a, under the Δ-partition Γ, determines an absolutely continuous motion $x[t] = x[t, t_0, x_0; U_a, \Gamma; V_\tau]$, which satisfies the initial condition $x[t_0] = x_0$ and, for almost all $t \in [t_0, \vartheta]$, satisfies the contingency

$$\dot{x}[t] \in \mathcal{F}_v(t, x[t]; \mu_{[\tau_i, x[\tau_i], \Gamma]}) \qquad (\tau_i < t < \tau_{i+1}), \tag{6}$$

where the symbol $\mathcal{F}_v(t, x, \mu)$ denotes the convex hull of the set of vectors

$$f_v = \int f(t, x, u, v) \mu(du), \qquad v \in \mathcal{V}. \tag{7}$$

We shall say that the strategy $U_a^0 \div \{\mu\}_{[t,x,\Gamma]}$ ensures the approach of the point $x[t]$ to the target $\mathcal{M}$ if, for any number $\varepsilon > 0$, one can indicate a number

$\Delta(\varepsilon) > 0$ such that every motion $x[t] = x[t, t_0, x_0; U_a^0, \Gamma; V_\tau]$ intersects the ε-neighborhood $\mathcal{M}_\varepsilon$ of the set $\mathcal{M}$ at least once for $t_0 \le t \le \vartheta$, whatever the Δ-partition Γ, whose

$$\Delta \le \Delta(\varepsilon). \tag{8}$$

Now the guidance game problem under consideration is formulated as follows.

Problem 1. Find a strategy U_a^0 which ensures the approach of the point $x[t]$ to the target $\mathcal{M}$.

The solution of this problem is determined by the following two circumstances, which follow from Theorems 1 and 2 given below.

Suppose that in the space $\{x\}$, on the interval $[t_0, \vartheta]$, a system of closed sets $\mathcal{W}(t)$ is given. The strategy $U_a^{(e)} \div \{\mu^{(e)}\}_{[t,x]}$, extremal with respect to this system of sets, is determined from the condition: if $\varepsilon \ge 0$ is the distance from the point x to the set $\mathcal{W}(t)$, then among the measures $\mu^{(e)}(du)_{[t,x]}$ specifying the strategy $U_a^{(e)}$, the ε-strategies (^{16}) corresponding to the game

$$\gamma = \max_\mu \min_\nu \left(s' \iint f(t, x, u, v)\mu(du)\nu(dv) \right) \tag{9}$$

are included, at least for one value of the vector s from the set $S(t, x)$, which is defined as the set of all vectors s of the form $s = x^0 - x$, where x^0 is a point of $\mathcal{W}(t)$ nearest (in the Euclidean metric) to the point x. (The superscript prime on s in (9) denotes transposition.)

We shall say that the sets $\mathcal{W}(t)$ $(t_0 \le t \le \vartheta)$ are u-stable if, whatever the values of $t_* < \vartheta$, a point $x_* \in \mathcal{W}(t_*)$, a strategy V_a, and a Δ-partition Γ (one of whose $\tau_j = t_*$), among the motions $x[t] = x[t, t_*, x_*; U_\tau; V_a, \Gamma]$ there is at least one motion $x[t]$ satisfying the condition $x[\tau_{j+1}] \in \mathcal{W}(\tau_{j+1})$ or the condition $x[t_*] \in \mathcal{M}$ for $t_* \in [\tau_j, \tau_{j+1})$.

Theorem 1. *If $x_0 \in \mathcal{W}(t_0)$ and the system of sets $\mathcal{W}(t)$ $(t_0 \le t \le \vartheta)$ satisfies the conditions $\mathcal{M} \subset \mathcal{W}(t)$, $\mathcal{W}(\vartheta) = \mathcal{M}$, and is u-stable, then the strategy $U_a^{(\varepsilon)}$, extremal with respect to this system of sets, solves problem 1.*

Theorem 1 raises the question of determining the maximally broad system of sets $\mathcal{W}(t)$ satisfying the conditions of this theorem. The answer to this question is determined by the following circumstance, which follows from the conditions of positional absorption of the target $\mathcal{M}$. We shall say that the process (1), (2), from the position $\{t_*, x_*\}$ $(t_* \le \vartheta)$, *absorbs the target $\mathcal{M}$* by time ϑ positionally if, whatever the strategies V_a, the Δ-partition Γ, and $\varepsilon > 0$, among the motions

$$x[t] = x[t, t_*, x_*; U_\tau, V_a, \Gamma]$$

there is at least one motion $x[t]$ intersecting the set $\mathcal{M}_\varepsilon$ at least once for $t_0 \leq t \leq \vartheta$. (Here it is convenient to restrict oneself to the class of strategies $V_a = \{v\}_{[t,x,\Gamma]}$ for which the sets $\{v\}_{[t,x,\Gamma]}$ are upper semicontinuous with respect to inclusion as x varies.) The set $\mathcal{W}(t, \vartheta)$ $(t \leq \vartheta)$ of all points x for which the process (1), (2) positionally absorbs the target $\mathcal{M}$ by time ϑ from the position $\{t, x\}$ will be called the *set of positional absorption of the target* $\mathcal{M}$ *by time* ϑ.

Theorem 2. *The absorption sets* $\mathcal{W}(t, \vartheta)$ $(t_0 \leq t \leq \vartheta)$*, which, of course, satisfy the conditions* $\mathcal{M} \subset \mathcal{W}(t, \vartheta)$ *and* $\mathcal{W}(\vartheta, \vartheta) = \mathcal{M}$*, are u-stable sets.*

Thus, from Theorems 1 and 2 there follows the following alternative, which corresponds to conditions of the type of saddle-point conditions in one or another differential game giving rise to problem 1 on approach.

Alternative. Either $x_0 \in \mathcal{W}(t_0, \vartheta)$, and then the first player can choose the strategy $U_a^0 = U_a^{(\varepsilon)}$, which ensures for him the approach of the motion $x[t]$ to the target $\mathcal{M}$; or x_0 is not contained in $\mathcal{W}(t_0, \vartheta)$, and then the second player can choose a strategy V_a which, for a suitable Δ-partition Γ, ensures for him the avoidance of the target $\mathcal{M}$ by the motion $x[t]$.

An example of an application of this alternative may be the conflict problem of the minimax of the time $T = \vartheta - t_0$ of approach of the point $x[t]$ to the set $\mathcal{M}$. Relying on the alternative, we arrive at the conclusion that, for the given initial position $\{t_0, x_0\}$, the desired minimax

$$T^0 = \vartheta^0 - t_0$$

is determined by the first absorption time ϑ^0, i.e., by the smallest value of ϑ for which

$$x_0 \in \mathcal{W}(t_0, \vartheta^0).$$

At the same time it follows that this problem has a saddle point in the class of mixed approximating strategies $\{U_a, V_a\}$.

The effective construction of the sets $\mathcal{W}(t, \vartheta)$, proceeding directly from their definition as sets of positional absorption, turns out to be difficult. These sets can be constructed by means of well-known transparent procedures (see, for example, ([1], [4–6], [8])), moving in steps Δt from the set $\mathcal{M}$ in the direction of backward time flow and then passing to the limit as $\Delta t \to 0$. In these procedures one need only, at each step, replace the enumeration of all possible pure programmed controls $v(\tau)$ $(\tau_{i+1} \leq \tau < \tau_i)$ by all possible mixtures of them, corresponding in their structure to the mixed strategies considered in this article. In this way one obtains another description of the same sets $\mathcal{W}(t, \vartheta)$. But even this description is difficult to use directly for the effective construction of the desired sets. The sets $\mathcal{W}(t, \vartheta)$, like the analogous absorption sets considered earlier in works ([12–15]), can be estimated with the aid of suitable sets of programmed absorption of the target $\mathcal{M}$ at the moment ϑ or by the moment ϑ. These sets of programmed absorption (with respect to mixed programs $v(\tau)$) sometimes admit a more effective description than the direct description of $\mathcal{W}(t, \vartheta)$. Criteria are constructed for them, indicating sufficient conditions under which such sets coincide with $\mathcal{W}(t, \vartheta)$. The methods of constructing these criteria repeat, with evident changes, the construc-

analogous criteria in the cases considered earlier (see, for example, ([15])). Finally, it should be said that under certain conditions, for example in the case when, for $\varepsilon > 0$, the set S appearing in condition (9) consists of a single vector s, the approximating extremal strategy $U_a^{(\varepsilon)}$, in the limiting form as $\Delta t = 0$, is formalized as a mixed strategy $U^{(\varepsilon)}$ operating within differential equations in contingencies, similarly to how this was done in the cases considered earlier ([12–15]).

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Note: Figure translations are in progress. See original paper for figures.

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