

SOLUTION OF AN ALGEBRAIC PROBLEM ENCOUNTERED IN CONTROL THEORY

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Abstract

Full Text

MATHEMATICS

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SOLUTION OF AN ALGEBRAIC PROBLEM ENCOUNTERED IN CONTROL THEORY

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1°. Suppose that a system of differential equations is given

$$dx/dt = Ax + b\xi \quad (1)$$

and a Hermitian form $\mathfrak{F}(x, \xi)$. In (1), A, b are complex or real constant matrices of orders $n \times n$, $n \times m$, and the vectors x, ξ have orders n, m , respectively. It is assumed that the system (1) is controllable, i.e., that the rank of the $n \times mn$ matrix $\|b, Ab, \dots, A^{n-1}b\|$ is equal to n . Consider the problem of representing the form $\mathfrak{F}(x, \xi)$ in the form*

$$\mathfrak{F}(x, \xi) = \frac{\partial}{\partial t}(x^* H x) + (\xi - h^* x)^* \Gamma (\xi - h^* x), \quad \forall x, \xi, \quad (2)$$

where $H = H^*, h, \Gamma = \Gamma^*$ are certain constant matrices of orders respectively $n \times n$, $n \times m$, $m \times m$, and the derivative is taken by virtue of system (1), i.e.

$$\frac{d}{dt}(x^* H x) = 2 \operatorname{Re}[x^* H (Ax + b\xi)].$$

It is obvious that the matrix Γ is determined at once from (2) by the relation $\mathfrak{F}(0, \xi) = \xi^* \Gamma \xi$, which we shall assume to be satisfied. The problem consists in determining conditions for the existence of the representation (2) and in finding a convenient procedure for determining matrices $H = H^*, h$ satisfying relation (2). The solution of this problem, for special $\mathfrak{F}(x, \xi)$, reduces to a number of problems in the theory of absolute stability (see, for example, ⁽¹⁾) and in the theory of optimal control (synthesis of optimal control in problems of minimizing quadratic functionals for linear stationary systems, i.e., in problems considered first by A. M. Letov—see, for example, ⁽²⁻⁵⁾). All these problems pertain either to the case $m = 1$, or to the case $m > 1$, $\Gamma \geq 0$ (for $m > 1$, $\Gamma \leq 0^{**}$).

There are many works in which, under various special assumptions, the indicated problem of the possibility of representation (2) is solved. The sign-definiteness property of the matrix Γ , or the condition $m = 1$, is then used essentially in the proofs.*** A number of problems in the theory of differential games lead to the

problem of representation (2) with $m > 1$, $\Gamma \not\geq 0$. Below a solution is given for the formulated problem under certain nonrestrictive assumptions excluding, in the space of coefficients of the form $\mathfrak{F}(x, \xi)$ and the system (1), a set of measure zero. The method of proof is substantially different from (1, 9) and others, and makes it possible to consider the case $\Gamma \geq 0$, which is not amenable to treatment by the known methods (1, 9) and others. It is important to note that for $m > 1$ the procedure obtained below for determining the matrices

* Here and below the asterisk denotes Hermitian conjugation. By I_n below is denoted the identity $n \times n$ matrix.

** The notation $\Gamma \geq 0$ ($\Gamma > 0$) means that $\Gamma = \Gamma^*$ and $\xi^* \Gamma \xi \geq 0$ for all ξ ($\xi^* \Gamma \xi > 0$ for all $\xi \neq 0$). The inequalities $\Gamma \leq 0$, $\Gamma < 0$ are defined analogously. The notation $\Gamma \not\geq 0$ means that the form $\xi^* \Gamma \xi$ assumes both positive and negative values.

*** For $m = 1$, solutions of related problems were first obtained in (6-8). For $m > 1$, $\Gamma \geq 0$, a solution of this problem under certain special assumptions (not, however, restricting generality) was given without proof in (1) (Theorem 3.4) and in the general case with proof in (9).

h, H is substantially more convenient than those previously known. For example, it does not require factorization of an $m \times m$ -matrix polynomial.

2°. Let λ be complex, ω a real variable, and let $X(\lambda)$ be some matrix or scalar polynomial $X(\lambda) = X_0 \lambda^N + \dots + X_N$. Below, by $X(\lambda)^\nabla$ we shall denote the polynomial $X(\lambda)^\nabla = [X(-\bar{\lambda})]^* = X_0^* (-\lambda)^N + \dots + X_N^*$. Obviously, $X(i\omega)^\nabla = X(i\omega)^*$, $[X(\lambda)Y(\lambda)]^\nabla = Y(\lambda)^\nabla X(\lambda)^\nabla$, $[\det X(\lambda)]^\nabla = \det[X(\lambda)^\nabla]$.

Introduce the following notation:

$$A_\lambda = \lambda I_n - A, \quad \delta(\lambda) = \det A_\lambda, \quad Q(\lambda) = A_\lambda^{-1} \delta(\lambda), \quad r(\lambda) = \left\| \begin{array}{c} Q(\lambda) \quad b \\ \delta(\lambda) \quad I_m \end{array} \right\|, \quad (3)$$

$$\mathfrak{F}(A_{i\omega}^{-1} b \xi, \xi) = \xi^* \Pi(i\omega) \xi, \quad \Phi(i\omega) = |\delta(i\omega)|^2 \Pi(i\omega).$$

Here $\Pi(i\omega) = \Pi(i\omega)^*$ is the matrix of order m of the Hermitian form $\mathfrak{F}(A_{i\omega}^{-1} b \xi, \xi)$. Since $\mathfrak{F}(0, \xi) = \xi^* \Gamma \xi$, we have $\Pi(\infty) = \Gamma$. From (3) we have

$$\xi^* \Pi(i\omega) \xi = \mathfrak{F}(\delta^{-1} Q b \xi, \delta^{-1} \delta \xi) = |\delta(i\omega)|^{-2} r(i\omega)^* F r(i\omega),$$

where $F = F^*$ is the matrix of the form \mathfrak{F} . Therefore

$$\Phi(\lambda) = r(\lambda)^\nabla F r(\lambda).$$

Thus, $\Phi(\lambda)$ is a matrix polynomial with leading term $(-1)^n \lambda^{2n} \Gamma$, and $\Phi(\lambda)^\nabla = \Phi(\lambda)$. Below, for brevity, the argument λ is sometimes omitted.

Lemma 1. The following formulas hold:

$$\det \Phi = \delta^{m-1} (\delta^\nabla)^{m-1} \det \Gamma \cdot \varphi, \quad \Phi^{-1} = \Omega / (\delta \delta^\nabla \varphi),$$

where φ, Ω are scalar and matrix polynomials;

$$\varphi = \varphi^\nabla;$$

$$\Omega = \Omega^\nabla, \quad \varphi = (-1)^n \lambda^{2n} [1 + O(\lambda^{-1})],$$

$$\Omega = (-1)^n \Gamma^{-1} \lambda^{2n} [I_m + O(\lambda^{-1})] \quad \text{as } \lambda \rightarrow \infty.$$

Let $\alpha(\lambda), \beta(\lambda)$ be polynomials. By $\text{rem}(\beta/\alpha)$ below is denoted the polynomial that is the remainder obtained upon division of $\beta(\lambda)$ by $\alpha(\lambda)$. If $B(\lambda) = \|\beta_{jk}(\lambda)\|$ is a matrix polynomial, then

$$\text{rem}(B/\alpha) = \|\text{rem}(\beta_{jk}/\alpha)\|.$$

By $\langle \alpha, \beta \rangle$ below is denoted the greatest common divisor of the polynomials $\alpha(\lambda)$ and $\beta(\lambda)$, and by $\langle \alpha, B \rangle$, the greatest common divisor of the polynomials $\alpha(\lambda)$ and all $\beta_{jk}(\lambda)$.

Theorem. Suppose that the system (1) is controllable and that $\det \Gamma \neq 0$.

I. Let (2) hold. Then: a) the polynomial $\varphi(\lambda)$ admits the factorization

$$\varphi(\lambda) = \psi(\lambda)\psi(\lambda)^\nabla,$$

where $\psi(\lambda) = \det(\lambda I_n - A - bh^*)$, and, consequently, either $\varphi(i\omega) = 0, \forall \omega$, or all purely imaginary zeros of the polynomial $\varphi(\lambda)$ have even multiplicity; b) the following is fulfilled

$$h^*q(\lambda) = \delta(\lambda)\Omega_0(\lambda), \quad \forall \lambda, \quad q(\lambda) = \text{rem}(Qb\Omega/\psi\delta), \quad \Omega_0(\lambda) = \text{rem}(\Omega/\psi). \quad (4)$$

II. Let

$$\langle \delta, \delta^\nabla \rangle = 1, \quad \langle \varphi, \delta \delta^\nabla \rangle = 1, \quad \varphi(i\omega) \neq 0, \quad \forall \omega.$$

Let $\psi(\lambda)$ be an arbitrary polynomial satisfying the factorization relation

$$\varphi(\lambda) \equiv \psi(\lambda)\psi(\lambda)^\nabla, \quad \forall \lambda$$

and such that

$$\langle \psi, \psi^\nabla \rangle = 1.$$

Let h be an arbitrary $n \times m$ -matrix satisfying the identity (4) or, otherwise, such that

$$(I_m - h^*A_\lambda^{-1}b)\Omega(\lambda)/\psi(\lambda)$$

is a polynomial*. Let $G = G^*$ be the matrix of the Hermitian form $\mathfrak{F}(x, 0)$ and let $H = H^*$ be the matrix determined from the linear system

$$HA + A^*H = G - h\Gamma h^*.$$

Then the identity (2) holds.

Remark. Suppose that the conditions

$$\langle \delta, \delta^\nabla \rangle = 1, \quad \langle \varphi, \delta \delta^\nabla \rangle = 1$$

are not fulfilled. Make in (1) the substitution

$$\xi = \xi_1 + a^*x,$$

where a is some $n \times m$ -matrix. Then we obtain a new system

$$dx/dt = A_1x + b\xi_1,$$

where

$$A_1 = A + ba^*,$$

and a new form

$$\mathfrak{F}_1(x, \xi_1) = \mathfrak{F}(x, \xi).$$

It can be shown that the new polynomial $\varphi(\lambda)$ coincides with the old one. With a suitable choice of a , the polynomial δ , as is known, can be made arbitrary (with leading term λ^n). Therefore, for the new system, with a suitable choice of a , the conditions

$$* \text{ It can be shown that } Qb\Omega = \delta q_1,$$

where q_1 is a polynomial. Therefore

$$q(\lambda) = \text{rem}(\delta q_1 / \delta \psi) = \delta q_0,$$

where

$$q_0 = \text{rem}(q_1 / \psi),$$

and the identity (4), from which the matrix h is determined, has the form

$$h^*q_0(\lambda) = \Omega_0(\lambda), \quad \forall \lambda.$$

$\langle \delta, \delta^\nabla \rangle = 1, \langle \varphi, \delta \delta^\nabla \rangle = 1$ will be satisfied. Thus, if representation (2) holds and $\varphi(i\omega) \neq 0, \forall \omega$, then it will be found by means of the indicated substitution and the use of the procedure indicated in part II of the theorem. We note that, after equating the coefficients of equal powers of λ , the equation $h^*q(\lambda) = \delta(\lambda)\Omega_0(\lambda)$ becomes a system of $2nm^2$ linear equations with respect to mn elements of the matrix h . It can be shown that, for $\varphi(i\omega) \neq 0, \forall \omega$, this system always has a solution. Therefore, for the existence of matrices h, H satisfying relation (2), it is sufficient that $\varphi(i\omega) \neq 0, \forall \omega$.

3°. Proofs.

Lemma 2. Let $\Sigma(\lambda) = c_0^*Q(\lambda)b_0 + \Gamma_0\delta(\lambda)$, where b_0, c_0, Γ_0 are arbitrary $n \times m$ -, $n \times m$ - and $m \times m$ -matrices. Then $\det \Sigma(\lambda) = \delta(\lambda)^{m-1}\varphi_0(\lambda)$, where $\varphi_0(\lambda)$ is a polynomial of degree not exceeding n .

Put

$$U = \begin{vmatrix} I_n; & -A_\lambda^{-1}b_0 \\ c_0^*; & \Gamma_0 \end{vmatrix}, \quad V = \begin{vmatrix} I_n; & A_\lambda^{-1}b_0 \\ 0; & I_m \end{vmatrix}, \quad W = \begin{vmatrix} A_\lambda + b_0c_0^*; & -b_0(\Gamma_0 - I_m) \\ c_0^*; & \Gamma_0 \end{vmatrix}.$$

We have $UV = \|R_{jh}\|$, $j, h = 1, 2$, where $R_{11} = I_n$, $R_{12} = 0$, $R_{22} = \delta\Sigma$.
Therefore

$$\det(\delta\Sigma) = \det(UV) = \det(VU).$$

Since $VU = \text{diag}(A_\lambda^{-1}, I_m) \cdot W$, it follows that

$$\det(\delta\Sigma) = \det A_\lambda^{-1} \det W.$$

Obviously, $\varphi_0 = \det W$ is a polynomial of degree $\leq n$.

Corollary. For $\Gamma_0 = I_m$ we obtain the well-known formula

$$\det(A_\lambda + b_0c_0^*) = \det A_\lambda \cdot \det(I_m + c_0^*A_\lambda^{-1}b_0).$$

Lemma 3. Every minor of order k ($k \leq m$) of the matrix $\Sigma(\lambda)$ defined in Lemma 2 is divisible by δ^{k-1} , and the quotient is a polynomial of degree not exceeding n .

Indeed, the matrix Σ' of the indicated minor has the form $\Sigma' = e_1^*\Sigma e_2$, where e_1, e_2 are constant matrices of order $m \times k$. Applying Lemma 2 to the matrix

$$\Sigma' = c_1^*Q(\lambda)b_1 + \Gamma_1\delta(\lambda),$$

we obtain the required assertion.

Lemma 4. Every minor of order k ($k \leq m$) of the matrix $\Phi(\lambda)$ is divisible by $\delta^{k-1}(\delta^\nabla)^{k-1}$, and the quotient is a polynomial of degree not exceeding $2n$.

Let

$$1 \leq j_1 < \dots < j_k \leq m; \quad 1 \leq h_1 < \dots < h_k \leq n; \quad j = (j_1, \dots, j_k), \quad h = (h_1, \dots, h_k);$$

Φ_{jh} be the matrix of the corresponding minor of the matrix Φ . From the formula $\Phi = r^\nabla Fr$ we have

$$\det \Phi_{jh} = \sum (\det r_{\mu j})^\nabla \det F_{\mu\nu} \cdot \det r_{\nu h},$$

where μ, ν are analogous sets of k indices, and $r_{\mu j}, F_{\mu\nu}$ are analogous $k \times k$ -submatrices of the matrices r, F . Obviously, $r_{\mu\nu}$ has the form

$$r_{\mu\nu} = c_{\mu\nu}^*Q(\lambda)b_{\mu\nu} + \Gamma_{\mu\nu}\delta(\lambda),$$

where $b_{\mu\nu}, c_{\mu\nu}, \Gamma_{\mu\nu}$ are certain constant $n \times k$ -, $n \times k$ - and $k \times k$ -matrices. By Lemma 2,

$$\det r_{\mu\nu} = \delta^{k-1}\varphi_{\mu\nu},$$

where $\varphi_{\mu\nu}$ is a polynomial of degree not exceeding n . Hence the assertion of Lemma 4 follows.

Lemma 5. *Let system (1) be controllable, $\langle \delta, \delta^\nabla \rangle = 1$, and let h be an arbitrary $m \times n$ -matrix satisfying the factorization relation*

$$\Phi(\lambda) = \Psi^\nabla \Gamma \Psi,$$

where

$$\Psi = \delta(\lambda)I_m - h^*Q(\lambda)b.$$

Let the matrix $H = H^*$ be determined from the equation

$$HA + A^*H = G - h\Gamma h^*,$$

where $G = G^*$ is the matrix of the form $\mathfrak{F}(x, 0)$. Then identity (2) is fulfilled.

Denote the form on the right-hand side of (2) by $\mathfrak{H}(x, \xi)$, put

$$\mathfrak{H}(x, \xi) - \mathfrak{F}(x, \xi) = \mathfrak{F}_1(x, \xi),$$

and show that $\mathfrak{F}_1(x, \xi) \equiv 0$. We have

$$\mathfrak{F}_1(x, 0) \equiv 0, \quad \mathfrak{F}_1(0, \xi) \equiv 0.$$

Therefore

$$\mathfrak{F}_1(x, \xi) = \operatorname{Re}(\xi^* f^* x),$$

where f is some $n \times m$ -matrix. The identity

$$\Phi(i\omega) = \Psi(i\omega)^* \Gamma \Psi(i\omega)$$

means that $\mathfrak{F}_1(x, \xi) \equiv 0$ for

$$Ax + b\xi = i\omega x.$$

Put

$$\chi(\lambda) = \xi^* f A_\lambda^{-1} b \xi,$$

where ξ is an arbitrary fixed complex vector. We have

$$\operatorname{Re} \chi(i\omega) = 0, \quad \forall \omega.$$

By the symmetry principle, the poles of $\chi(\lambda)$ are located symmetrically with respect to the imaginary axis. On the other hand, they can only be eigenvalues of the matri-

trices A , among which there are none situated symmetrically with respect to the imaginary axis, by virtue of the condition $\langle \delta, \delta^\nabla \rangle = 1$. Therefore $\chi(\lambda)$ has no poles. Consequently, $\chi(\lambda) = \text{const}$, and, since $\chi \rightarrow 0$ as $\lambda \rightarrow \infty$, we have $\chi(\lambda) \equiv 0$. In view of the arbitrariness of ξ , we have $f^* A_\lambda^{-1} b = 0$. From the controllability of (1) it follows that $f = 0$, $\mathfrak{F}_1(x, \xi) = 0$.

Proof of Lemma 1. From Lemma 4 it follows that φ is a polynomial. Since $\Phi = (-1)^n \lambda^{2n} \Gamma + \dots$, it follows that $\det \Phi = (-1)^{mn} \lambda^{2mn} \det \Gamma \times [1 + O(\lambda^{-1})]$. Hence $\varphi = (-1)^n \lambda^{2n} [1 + O(\lambda^{-1})]$.

Let $\Omega = \Phi^{-1} \delta \delta^\nabla \varphi$. We have* $\Omega = \Phi_{\text{pr}} / [\delta^{m-2} (\delta^\nabla)^{m-2} \det \Gamma]$. By Lemma 4, Ω is a polynomial. From (3), $\Omega = \Pi^{-1} \varphi$. Therefore, as $\lambda \rightarrow \infty$, we have $\Omega / \lambda^{2n} \rightarrow (-1)^n \Gamma$, i.e. $\Omega = \lambda^{2n} (-1)^n \Gamma [1 + O(\lambda^{-1})]$. The relations $\varphi = \varphi^\nabla$, $\Omega = \Omega^\nabla$ follow from $\Phi = \Phi^\nabla$.

Proof of the theorem. I. Denote $\Psi(\lambda) = \delta I_m - h^* Q b$. According to the corollary to Lemma 2, $\det(\lambda I_m - A - bh^*) = \delta \det(I_m - h^* A^{-1} b) = \delta \det(\delta^{-1} \Psi)$. Therefore $\det \Psi = \delta^{m-1} \psi$, where $\psi = \det(\lambda I - A - bh^*)$. For $Ax + b\xi = i\omega x$, from (2) we have $\mathfrak{F}(A_{i\omega}^{-1} b \xi, \xi) = \xi^* \Psi(i\omega)^* \Gamma \Psi(i\omega) \xi / |\delta(i\omega)|^2$. From (3), $\mathfrak{F}(A_{i\omega}^{-1} b \xi, \xi) = \xi^* \Phi(i\omega) \xi / |\delta(i\omega)|^2$. Therefore $\Phi(i\omega) = \Psi(i\omega)^* \times \Gamma \Psi(i\omega)$, $\Phi(\lambda) = \Psi(\lambda)^\nabla \Gamma \Psi(\lambda)$, $\det \Phi = \delta^{m-1} (\delta^\nabla)^{m-1} \psi \psi^\nabla \det \Gamma$. Comparing with the expression for $\det \Phi$ in Lemma 1, we obtain $\varphi = \psi \psi^\nabla$, whence assertion a) follows. By Lemma 3, $\Phi_{\text{pr}} = \delta^{m-2} Z$, where Z is a polynomial. From $\Phi = \Psi^\nabla \Gamma \Psi$ we have $\Phi_{\text{pr}} = \Psi_{\text{pr}} \Gamma_{\text{pr}} \Psi^\nabla$, $\Psi \Phi^{-1} \det \Phi = \Gamma_{\text{pr}} \Psi^\nabla \det \Psi$. Substituting into the latter equality the values of Φ^{-1} , $\det \Phi$ from Lemma 1, and also $\det \Psi = \delta^{m-1} \psi$ and $\Psi^\nabla = \delta^{\nabla m-2} Z_{\text{pr}}$, we obtain $\Psi \Omega = \Gamma^{-1} Z^\nabla \delta \psi$, $h^* Q b \Omega \equiv \delta \Omega \pmod{(\psi \delta)}$. This implies (4), i.e. c).

II. Let h satisfy (4) and $\varphi = \psi \psi^\nabla$. Put again $\Psi = \delta I_m - h^* Q b$. From (4) we obtain $\Psi \Omega = Z_0 \varphi \delta$, where Z_0 is a matrix $m \times m$ polynomial. Denote $\langle \psi, Z_0^\nabla \rangle = \psi_0$, $Z_0^\nabla = Z_1 \psi_0$, $\psi = \psi_0 \psi_1$, where Z_1, ψ_1 are polynomials, $\langle \psi_1, Z_1 \rangle = 1$. Since $\Omega = \Omega^{\nabla \nabla}$ (Lemma 1), from $\Psi \Omega = Z_1^\nabla \psi_0^\nabla \psi \delta$ we have $\Psi \Omega \Psi^\nabla = \Psi Z_1 \psi_0 \psi \delta^\nabla$. Substituting the values of $\Psi \Omega$ and $\psi = \psi_0 \psi_1$, we obtain $Z_1^\nabla \Psi^\nabla \psi_1 \delta = \Psi Z_1 \psi_1 \delta^\nabla$. By the condition $\langle \psi_1, \psi_1^\nabla \rangle = 1$, $\langle \psi_1, \delta^\nabla \rangle = 1$, $\langle \delta, \delta^\nabla \rangle = 1$. Therefore $\Psi Z_1 = Y \psi_1 \delta$, where Y is a polynomial, and $Y^\nabla = Y$. Consequently, $\Psi \Omega \Psi^\nabla = Y \psi \psi^\nabla \delta \delta^\nabla$. Since $\deg \Omega = 2n$ (Lemma 1), $\deg \Psi = \deg \psi = \deg \delta = n$, we have $\deg Y = 0$, i.e. $Y = \text{const}$. Since, as $\lambda \rightarrow \infty$, $\Psi / \delta \rightarrow I_m$, $\Omega / \varphi \rightarrow \Gamma^{-1}$, it follows that $Y = \Gamma^{-1}$, and $\Psi \Omega \Psi^\nabla = \Gamma^{-1} \varphi \delta \delta^\nabla$. Passing in this relation to inverse matrices and substituting $\Omega^{-1} = \Phi / \delta \delta^\nabla \varphi$, we obtain that Ψ satisfies the factorization relation $\Phi = \Psi^\Delta \Gamma \Psi$. Applying Lemma 5, we obtain the assertion of the theorem.

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- * By Φ_{pr} here and below is denoted the adjugate matrix: $\Phi_{pr} = \Phi^{-1} \det \Phi$. By $\deg \Phi$ below is denoted the degree of the polynomial $\Phi = \Phi(\lambda)$.
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